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STATICS

BY

ALGEBRAIC AND GRAPHIC METHODS

INTENDED PRIMARILY FOR STUDENTS OF ENGINEERING AND ARCHITECTURE

BY

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PREFACE.

THIS volume has been written in the hope of helping students of engineering and architecture to acquire a knowledge of Statics which will include the power to apply it correctly in professional work. To this end an attempt has been made to carry out several specific purposes, prominent among which may be mentioned the following:

1. To give much attention to the starting points of the science, and to make as clear as possible the course of deduction therefrom.

2. To point out the inherent mathematical limitations of pure Statics, and to show how all its important problems are solved.

3. To develop algebraic and graphic methods of solution, or, if one prefer the terms, Analytical and Graphical Statics, side by side and with equal thoroughness.

4. To present a graded set of problems illustrating not only universal principles but also how Statics is used in engineering practice.

5. Finally, to keep the book of a size commensurate with the small amount of new matter which the reader, versed in the simplest operations of elementary mathematics, need master to gain the desired end.

It may be pointed out that the phrases Analytical Statics and Graphical Statics are avoided. The ground for this is that there seems to be no necessity for using them, in this work at

least, and that the terms seem objectionable as tending to obscure the unity of Statics, and to produce the impression that two merely alternative methods of procedure from identical premises and of identical mathematical significance are loosely connected if not actually distinct branches of the science.

The subject will be found developed in the following pages in such a way as to make it possible to solve problems from the outset by both methods in parallel (as in the plates), and the practice of making such double solution is believed to be of great value not only for the drill of checking one's own work, but also for the clearer light in which each method is seen by being kept in close relation to the other. Moreover, as the student checks the correctness of his own work, it is possible, even with classes of upwards of one hundred, to assign different problem data to each student, and still keep the labor of inspecting students' work at a minimum. After some experience with the two methods side by side, it is believed that practice should be had in the rapid solution of a large number of varied problems such as can be found in the familiar works on Statics, in which a single solution by either method is accepted and in which the student judges the correctness of his results not only by examination of his work, but by reflection upon their reasonableness under the conditions.

Of course many different sources have been drawn upon freely for suggestions, methods, and material, but it will not be out of place to mention as especially prominent among them Rankine's *Applied Mechanics* and Hoskins' *Graphic Statics*, and the writer takes pleasure in acknowledging his obligation accordingly. Grateful recognition is also due to Professor I. N. Hollis for valued criticism.

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PART I.

GENERAL PRINCIPLES AND METHODS.

INTRODUCTION.

1. **Algebraic and Graphic Methods Compared.**—There are two methods of representing quantities for the purpose of computation, one in which numerals or letters are used—the algebraic or so-called analytical method—and the other in which quantities are represented by the lengths and relative positions of lines—the graphic method.

The quantities of higher mathematics, such as imaginaries and infinitesimals, can be treated algebraically only. On the other hand, many complicated relations, such as profiles of railroad lines and the varying pressure in a steam cylinder, can be comprehended and made subjects of computation or measurement only by graphic representation.

The great bulk of ordinary arithmetic and trigonometric calculation can be done by either method, and which of the two should be selected is purely a matter of expediency. In the fundamental operations of arithmetic—addition, subtraction, multiplication, division, involution, and evolution—the graphic method is either so simple and obvious that it is rarely thought

of as a formal method in connection with them, or else so disadvantageous as to be little used.

Problems involving much finding and summing of products or involving trigonometric or complicated geometric relations are found to be an advantageous field for the use of graphic methods, and in this field, graphic methods assume a place of relative importance.

Problems in equilibrium are of precisely this sort, and graphic methods of dealing with them have been developed very extensively, and even named, as a body, Graphical Statics, in distinction from the body of algebraic methods in statical problems—called Analytical or Algebraic Statics. All these terms, as pointed out in the preface, seem objectionable to the writer and will not be used further in this work. The terms algebraic method and graphic method will be used in their stead.

The underlying principles and the procedure in both methods are, of course, scientifically correct, but they differ in the degree of precision attainable. The precision of results in algebraic computation depends upon the extent to which decimals or significant figures are carried out, and in graphical work upon care and skill in draughting. In the former case absolute precision may be approached as closely as the computer may care to go, but in the latter early bounds are set by the limitations of draughting appliances and human eyesight. Sometimes it is worth while to use a combination of the two by sketching more or less roughly the desired diagrams and calculating trigonometrically instead of scaling the lengths of resulting lines. This of course is the algebraic method itself so far as precision goes.

Whatever may be said about their relative precision and scope, it is certainly true that neither method can yield a result more accurate than the data from which it is derived. In

structural design, the data are rarely or never known to a degree of accuracy greater than the degree of precision readily obtained in graphic computation, so that actually, from the point of view of the designing engineer, one method may be said to yield results as trustworthy as the other. As already stated, the two methods are of equal correctness in principle.

The choice between one method or the other for a given problem before an engineer is a matter of convenience, but in many cases both should be used for the sake of the check upon results thus obtained.

The graphic method has the advantage, sometimes important, that it presupposes a knowledge of little or no mathematics beyond the elements of plane geometry, while the draughting appliances required in its use are of the simplest.

2. History of Graphic Methods in Statics.—Knowledge of the graphic methods in statics and of their importance and scope is due mainly to the efforts of the late Professor Culmann of Zurich. Prior to 1866, when he published his great work, "Die Graphische Statik," the subject had attracted little attention and there existed only scattered and fragmentary writings on it. Culmann died after completing the first, only, of his contemplated volumes, but he had thoroughly established the subject, and interest in the graphic treatment of statical problems steadily spread through the engineering world. Among other names to be prominently associated with the development of the graphic methods are Bow, Maxwell, Mohr, W. Ritter, Cremona, Müller-Breslau, Lévy.

CHAPTER I.

DEFINITIONS AND PRELIMINARIES.

3. **Mechanics** is the science which treats of rest and motion. It is subdivided into **Kinematics** and **Dynamics.**

Kinematics treats of motion apart from its causes. Its scope is accordingly limited to the consideration of the paths of moving bodies and of that aspect of velocity and acceleration which involves only the relations between space and time.

Dynamics treats of the effect of forces upon rest or motion. It is subdivided into Kinetics and Statics. **Kinetics** deals with the cases in which given forces produce a change in a body with respect to rest or motion, and **Statics** with the cases in which no such change is produced.

Kinetics accordingly deals with the velocity and acceleration of given bodies resulting from given forces, and with such topics as work and energy, momentum, centrifugal force, impact, etc. Statics is concerned exclusively with the conditions under which a body under the action of forces will remain at rest or undergo no change of motion.*

Statics—henceforth the exclusive subject of this book—is accordingly the science of equilibrium, of stability—the science by the aid of which are determined the forces necessary to

* The four terms Kinematics, Dynamics, Statics, and Kinetics have an etymological fitness which is worth noticing. Kinematics is from Gr. *kinema*, motion; Dynamics from Gr. *dynamis*, force; Kinetics from Gr. *kinetikos*, putting in motion; and Statics from Gr. *statikos*, causing to stand.

maintain a body undisturbed in its rest or motion in spite of disturbing tendencies, the forces which must interact between the various parts of the body to prevent its disruption under the action of given forces, and moreover the position in which a body or system of bodies under a given set of forces will be at rest.

4. Rigid Body. Particle.—A **rigid body** is a body conceived to be incapable of change in shape or size under the action of forces. Such a body could be affected only as a whole by forces which act upon it. Hence to state that the body concerned in any problem is rigid or is to be so considered is equivalent to stating that no thought is expected to be given to the possibility of the problem being complicated by the deformation or rupture of the body.

Although no such thing as a rigid body exists in nature, all solids approximate rigidity to a greater or less extent. In the case of solids subject only to forces well within their capacity to withstand them, such as all properly designed engineering structures, the approximation to rigidity is very close, and for the purposes of statics no material error results from assuming complete rigidity in such cases.

A complete examination into the safety of an engineering structure involves (*a*) the determination of the forces acting on the body or transmitted through its various parts; and (*b*) the study as to whether the body and its parts are able to resist such forces without rupture or undue deformation. In the first step the body is assumed rigid, because the results of the step are practically the same as if the body were rigid, and in the second the lack of rigidity is clearly recognized and provided against. The first step falls within the domain of statics, and the second within that of the kindred science, Resistance of Materials.

In statics it is the general practice to treat bodies as if they

were rigid, unless distinctly stated to the contrary,—actual lack of rigidity being taken into account only as an aid in the solution of certain complex problems which would otherwise be indeterminate.

A **particle** is a rigid body of the smallest conceivable dimensions. It is also called a **material point,** or, for short, simply a **point.**

Every problem involving forces implies the existence of bodies or particles on which the forces act.

5. **Rest and Motion.**—**Rest** is the relation existing between two points when a line connecting them does not change either in length or in direction. If, on the other hand, the line does change in either length or direction, the relation between the points is **motion.** This line can change in one or both of these two particulars only, and motion can accordingly be divided into two corresponding classes only, **translation** and **rotation.**

If the line between two points changes in length, one point is in translation with regard to the other. If the change is in direction, one point is in rotation about the other.

Two bodies are at rest with regard to each other when every point of one is at rest with regard to every point of the other.

Two bodies are in motion with regard to each other when any point of one is in motion with regard to any point of the other.

The motion of a body is **translation** when any point of that body describes a straight line. If the points of the body describe a series of straight lines, the motion is **pure translation.**

The motion of a body is **rotation** when such points as change in position at all describe sets of concentric circles in parallel planes.

The common normal to these planes which contains the

centers of all the circles is called the **axis of rotation.** This axis may or may not traverse the body, but any points not changing in position must be contained in it. If the axis of rotation is fixed in position, the motion is **pure rotation.**

Of course there can be compound translation, as that of a body sliding across the deck of a moving ship; or compound rotation, as that of the moon about its own center, and also about the earth; or combined translation and rotation, as that of a ball rolling in a straight path.

Two bodies at rest with reference to each other may be moving with respect to a third. Rest, then, far from meaning absolute motionlessness, means only motionlessness with regard to a definite body of reference, which in our work will usually be understood to be the earth.

It may be added that translation can be regarded as the extreme case of rotation in which the axis is at an infinite distance, and instead of two subdivisions of motion, rotation alone might be regarded as covering the whole ground. This point of view, however, will not be convenient for the purpose now in hand.

6. Force.—Force is an interaction between two bodies, either causing or tending to cause a change in their relative rest or motion. A force may always be conceived of as a push or a pull.

Forces are designated by single capital letters as P, Q, R, etc. In graphical work more elaborate designations are needed, which will be explained in Chapter II.

7. Elements of a Force.—In order that a force may be fully known, three characteristics of it, which will be spoken of as **elements,** are essential and sufficient, viz.:

1. **Magnitude.**
2. **Direction** (including sense).
3. **Point of Application.**

8. Magnitude.—The **magnitude** of a force is given by stating numerically the ratio of its effectiveness in producing translation to that of some arbitrarily selected standard force, usually the force of gravitation on a certain mass of standard material at a certain level and latitude; i.e., to the amount of force exerted by gravitation on the standard pound or other standard weight. In other words, the magnitude of a force is stated in units of weight.

The absolute unit of force, that is the force which, acting for a unit length of time on a unit mass of material, will give rise to a unit velocity, is not used in statics.

9. Direction and Sense.—The **direction** of a force is the direction of the line along which the force tends to produce motion, i.e., of the **line of action** of the force. Direction includes the sense of the force as well as the slope of its line of action. By **sense** is to be understood the specification as to which of the two ways along the line of action the force tends.

Sense is the distinction between forward or back, up or down, to right or left on a given line. It is merely a matter of algebraic sign, requiring no separate equation for its determination, and hence is not ranked as an element.

10. Point of Application.—The **point of application** is the place (treated as a point) upon a body where the force is brought to bear. This point is, of course, upon the line of action and, together with the slope, it locates that line.

In the case of rigid bodies, any point whatever on the line of action of the force may be taken as the point of application.*

Change in the point of application of a given force affects the body only as to rotation, not at all as to translation.

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* This will be seen upon adding anywhere in the line of action of the force two opposite forces coincident with it and both equal to it in magnitude. The three, obviously equivalent to the original, amount simply to a new one just like it.

11. Coplanar and Non-Coplanar Forces.—Forces are spoken of as **coplanar** and **non-coplanar** according as their lines of action do or do not lie in one plane.

Except where distinctly stated to the contrary, and except where statements are evidently general (as in the whole of this chapter), coplanar forces will form the exclusive subject of these pages. This limitation conduces to simplicity without real loss of generality, for coplanar forces once understood, the treatment of non-coplanar forces will offer little difficulty to the unaided reader.

12. Concurrent and Non-Concurrent Forces.—Forces are spoken of as concurrent or non-concurrent according as their lines of action do or do not meet in a point. Of course, either may or may not be coplanar. The study of concurrent forces is sometimes spoken of as statics of a particle, and of non-concurrent forces as statics of a rigid body.

Whether a set of forces is concurrent or not is a matter affecting rotation only, the translatory effect of a force being independent of its point of application.

13. Equilibrium.—Equilibrium is that condition of a set of two or more forces the result of which is that their combined effect on a body produces no change in the body with respect to rest or motion.

A body is said to be in equilibrium when it is acted on by a set of forces in equilibrium. A body in equilibrium may be either at rest or moving with uniform speed (either of translation or rotation) with regard to the body of reference. If it be moving in either of the ways just mentioned, it is the result of the previous application of an unbalanced force or of unbalanced forces, and any forces actually affecting it while in such motion are forces under the action of which a body could equally well be at rest.

A set of forces, therefore, under which a body will be at

rest is always a set in equilibrium; or, in other words, conditions of rest are always conditions of equilibrium. It follows that, by establishing the conditions of rest, the familiar most usual phase of equilibrium, we establish the general, universal conditions of equilibrium.

14. Equivalence.—Two forces or sets of forces having identical effects on the rest or motion of a body both with respect to translation and to rotation are termed **equivalent.**

15. Resultant and Equilibrant. Composition of Forces.—A single force equivalent to a set of forces is called the **resultant** of the set. A single force the addition of which to a set of forces produces equilibrium is called the **equilibrant** of that set of forces. The resultant and the equilibrant for a given set differ in sense only, i.e., one is simply the equal and opposite of the other. Of a set of forces in equilibrium any one of them is the equilibrant of all the rest. The operation of finding the resultant or equilibrant of a set of forces is called **Composition of Forces.**

16. Components.—Any one of a set of forces having a given resultant is a **component** of that resultant. Evidently there may be any number of components of a force.

Rectangular components of a force are two components equivalent to it whose lines of action are at right angles to each other.

A component in a given direction is understood to be the one of a pair of rectangular components whose line of action is parallel to the given direction. Such a component measures the total tendency of the given force to produce motion in the given direction.

17. Relation between Two Components and their Resultant. Parallelogram of Forces.*—If the adjacent sides of a parallelogram represent two concurrent forces in magnitude and

* See also § 29 for proof and additional discussion of the parallelogram of forces.

direction, the diagonal of this parallelogram starting at the intersection of these two sides will represent their resultant in magnitude and direction, *provided* the two forces act either both toward or both away from this intersection. The resultant will then also act toward or away from the point of intersection as the case may be. Ordinarily the components are regarded as acting away from their common point, and accordingly the resultant also.

Corollary.—If α be the inclination of any force, P, to any given line, the components along and at right angles to this line will have the magnitudes $P \cos \alpha$ and $P \sin \alpha$ respectively.

EXERCISE 1.* COMPONENTS. Find the components, both algebraically and graphically, of a force P whose magnitude, direction, and point of application are 20 lbs., 210°, and (2, 6) respectively, along the axes of X and Y, and also along five lines, $M_1 N_1$, $M_2 N_2$, $M_3 N_3$, $M_4 N_4$, $M_5 N_5$, inclined to the axis of X by 45°, 60°, 120°, 45°, and 30° respectively, and which pass through points (0, 0), (−6, 0), (6, 0), (8, 0), and 10, 0) respectively.

SUGGESTIONS AS TO GRAPHIC WORK.

Use as scales 1 in. = 10 lbs. and 4 units of length. Plot the line of action of P in its proper relation to the preferably horizontal and vertical axes of coordinates and the given lines $M_1 N_1 \ldots M_5 N_5$. This will serve as a diagram of data for both methods. The problem can now well be stated in condensed form across the top of the sheet.

Show P apart from this diagram with the proper direction and with the magnitude marked off on it to scale. At a little distance from it, draw parallels to all the given lines. Construct the required components, scale, and show results in their proper places with dimension lines.

SUGGESTIONS AS TO ALGEBRAIC WORK.

Determine the angles between the line of action of P and each of the seven given lines. Proceed with the necessary computations using this form:

$$\text{For } M_1 N_1, \quad 20 \overset{0.707}{\cos} 45° = 14.14.$$
$$\text{" } M_2 N_2, \text{ etc.}$$

* For the solution of this and all the following exercises, sheets of paper about 12×18 inches, preferably ruled in inch and tenth-inch squares, are recommended as affording sufficient room, and as convenient in shape for the double solutions usually required.

Finally collect results and show them in parallel columns thus:

Line	Gr.	Al.
$M_1 N_1$	14.1	14.14

QUESTIONS ON EXERCISE 1.

How does the line M_1N_1 differ from M_4N_4?

What difference does this cause in the components along these two lines?

How would changing the point of application of P to say $-3, 5$, or elsewhere, affect the results of this exercise?

From the seven components found, can two be selected which would fully replace P if properly located? Is more than one such selection possible?

What would have to be the location of such components in order that they might replace P?

18. Couples and their Moments.—Two equal and opposite forces with parallel lines of action constitute a **couple.** The sole tendency of a couple is to produce rotation. The measure of a couple, which can be nothing else than the measure of its tendency to produce rotation, is called the **moment** of the couple.

It has been observed as one of the fundamental laws of matter that the moment of a couple is proportional alike to the common magnitude of the forces of the couple and to the shortest distance between their lines of action. This distance is called the **arm** of the couple. The product of such magnitude and arm expresses the moment of the couple.

The **unit couple** consists of two forces of unit magnitude with unit arm, and its moment is the unit for measuring couples. This unit is called, in the English system, inch-pound, foot-pound, foot-ton, etc. It must not be confounded with the units of work and energy with the same names.

The sense of a couple will be called positive or negative according as its tendency is clockwise or not.

The plane of a couple once known nothing further need be

known of it but its moment, including, of course, the sense of the possible rotation.

Either of the forces of a couple can be treated for computation exactly like one of any other system of forces.

The effect of a couple is the same wherever in its plane or in a parallel plane its forces may be applied.

Couples in the same plane can be summed by summing their moments, and a couple equivalent to the set can be thus evaluated, just as forces in the same straight line can be summed by summing their magnitudes with the effect of evaluating a force equivalent to the set. In such work sense must, of course, be duly regarded.

19. Center of Rotation for a Couple.—The center of rotation for a couple may be anywhere in its plane, and is determined only by the relative motions of the bodies from which come the forces constituting the couple.

An interesting and important special case is that in which only one of the forces can be maintained in action at successive points of application. The center of rotation will then be the point of application of the other force, a familiar illustration being a pulley and belt. The point of application of the resistance of the shaft cannot change, while that of the force from the belt retreats freely as the pulley turns. The center of rotation is accordingly at the center of the shaft.

20. Single Force Always Replaceable by a Single Force and a Couple.—Let P (Fig. 1) be any force applied to a body and m be any point in the body. Suppose two additional forces, P' and P'', opposite in sense, parallel to P and of the same magnitude as P, to be applied to the body at m, distant p from the line of action of P. Evidently the condition of the body is unchanged, but the three forces

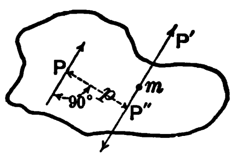


FIG. 1.

are seen to be equivalent to P with a new point of application m, and a couple of value $P \times p$.

Hence it may be stated that any force P can be replaced by a force of the same magnitude and direction distant p from its original position, and a couple of the value $P \times p$. In other words, points of application of forces can be shifted at will without effect on equilibrium, *if*, at each change, a suitable couple be added to the system.

Any set of non-concurrent forces can thus be reduced to an equivalent set of couples and concurrent forces.

21. Moment of a Force.—The **moment of a force** with respect to a point is the name given the moment of the couple consisting of the given force, and a force equal, opposite, and parallel to it conceived to be acting at the given point.*

The arm of this couple is evidently the perpendicular distance from the point to the line of action of the given force, a distance which may conveniently be called the **arm of the force** with respect to the point.

The moment of a force with respect to a point is accordingly expressed by the product of the magnitude of the force into its arm, and is called positive or negative according as the conceived tendency to rotation is clockwise or not.

The point to which the moment is referred is commonly called the **center of moments**, or, more briefly, the **center.**

The moment of a force is sometimes referred to an axis, meaning thereby the axis of the conceived rotation. This axis will be the straight line through the center normal to the plane of the force and its arm. The moment with respect to this

* Note that this is the same thing as saying that the moment of a force with respect to a point is the measure of its tendency to produce rotation about the point, *assuming the point to be fixed.* For assuming the point to be fixed is really assuming it to be always subject to a force equal, opposite, and parallel to the given one. This point of view gives rise to the phrase moment of a force *about* a point.

axis is also spoken of as the moment with respect to the plane which contains this axis and is parallel to the force.

EXERCISE 2. MOMENTS OF A FORCE.—Find the moments in foot-pounds, both in magnitude and sign, of a force of 20 lbs., whose direction and point of application are respectively 60° and (6, 2), about the five points (0, 0), (−3, −6), (8, 0), (−5, 2), (4, −3).

a. Obtain the results in the most direct way, by scaling arms from a carefully plotted diagram of data. Note that the result necessarily lacks precision and is affected by errors in draughting. Scale 1 in. = 2 ft. Show scaled dimensions in the diagram.

b. Obtain the results by adding algebraically obtained moments of components of the given force—a purely algebraic process. Note that the results are wholly independent of draughtsmanship and make a rigid check of results of (*a*) and can be made as precise as the nature of the case permits.

Do the work in this form:

$$a\begin{cases} M_1 = 20 \times ? = ? \\ M_2 = 20 \times ? = ? \end{cases}$$

etc.

$$b\begin{cases} M_1 = 20 \underset{10.0}{\overset{0.500}{\cos 60°}} \times 2 - 20 \underset{5.196}{\overset{0.866}{\sin 60°}} \times 6 = 20.00 - 103.92 = -83.92 \\ M_2 = \quad \text{etc.} \end{cases}$$

Record results in this form:

	a	*b*
M_1		
M_2		
M_3		

22. Propositions Regarding the Moment of a Force.—There follow at once from the definition of the moment of a force the three following important propositions.

I. For a force greater than zero, the moment can be zero only with respect to a point in its own line of action.

II. A zero moment with respect to a point off a stated line of action can result only from a zero force in that line.

III. A zero force with an infinite arm can have a finite moment.

The "zero force with an infinite arm" of the last proposition is simply a way of expressing any couple, as will be seen later.

23. Proposition.—The sum of the moments of two equal, opposite, and parallel forces (i.e., the moment of a couple) is constant for all points in their plane.

This the reader can readily prove for himself by taking any center and summing for it the moments of the two forces, using any convenient letters for their arms.

24. Sets of Forces Classified.—It follows from § 5 that all possible conditions of a body with respect to rest and motion fall within one of four classes, viz.:

1. Translation.
2. Rotation.
3. Both Translation and Rotation.
4. Neither Translation nor Rotation.

Any set of forces must accordingly be such as will produce one of these four conditions. We know by observation that a single force will produce translation and that a couple will produce rotation. Knowing also that translation and rotation can also result from sets of forces, we unhesitatingly conclude that for producing translation, a set could be replaced by some single force and for producing rotation, by some couple. Translation being the result of a single force or a set equivalent thereto, and rotation the result of a couple or a set equivalent thereto, it follows that any set of forces must fall into one of four classes corresponding to one of the four conditions of a body mentioned at the outset of this section.

Any set of forces is therefore reducible to one of the following equivalents, viz.:

1. Single Force,
2. Couple,
3. Single Force and Couple,
4. Neither Single Force nor Couple,

corresponding respectively to the four conditions of a body just mentioned.

Of course, the last named of these conditions is equilibrium.

It should be pointed out, in passing, that a force and a couple in the same or parallel planes are reducible to a single force, and that the non-reducible case of a force and a couple is that in which the force is not parallel to the plane of the couple.

25. Conditions of Equivalence.—THEOREM.—If, for two sets of forces in a given plane, the sum of the moments of the forces about three points not in the same straight line be the same for each set, the two sets are equivalent.

PROOF.—As shown in the preceding section, each of the two sets must be reducible to a single force, a couple, or is in equilibrium, there being no possibility of the non-reducible case of a force and couple since the forces are coplanar. There can accordingly be but six different combinations of sets from this point of view. They and their respective relations to the theorem are as follows:

(*a*) *Each set reducible to a single force.*—Upon consideration of the moment of a force, as defined in § 21, it is apparent that two forces can have the same moment about a point only when the point is distant from their lines of action inversely as the magnitudes of the forces. In a given case any point outside a definite straight line cannot meet these conditions, except in the special case when the two forces are coincident, and of the same magnitude and direction, i.e., when the forces are equivalent. Therefore equality of moments for each of three points not in the same straight line demonstrates equivalence for this combination of sets.

(*b*) *Each set reducible to a couple.*—The moment of a couple (§ 23) being a constant for all points in its plane, equality of moments for the two sets for any one point demonstrates equivalence for this combination.

(*c*) *Each set in equilibrium.*—The two sets are equivalent by definition, and the sums of their moments are bound to be equal for any and all points. The constant value of this sum is of course zero.

(*d*) *One set reducible to a single force,* the other to a couple,—necessarily non-equivalent.—As in combination I., the moments here could be equal only for points on a definite straight line. Their being equal for each of three points not in a straight line would exclude the possibility of this combination.

(*e*) *One set reducible to a simple force, the other in equilibrium,*—necessarily non-equivalent.—Equality of moments could then arise only for points on a definite straight line—the line of action of the single force. Equality for each of three points not in a straight line would exclude the possibility of this combination.

(*f*) *One set reducible to a couple, the other to equilibrium,* —necessarily non-equivalent.—Here equality of moments could exist for no point whatever. Equality of moments for all points or any point excludes the possibility of this combination.

Since in all of the possible combinations of sets of forces, the equality of the sums of the moments of the forces of the two sets for three points not in the same straight line is impossible without equivalence in the sets, the theorem is proved.

Corollary.—If the sets are equivalent, their moments are equal for all points in the plane and conversely.

CHAPTER II.

NOTATION AND CONVENTIONS.

26. Notation. Conventions Regarding Elements of Forces. (For illustrations see § 28.)—Magnitude is expressed in **algebraic** work by numerals or letters; in **graphic** work by lengths of lines. The capitals P, Q, R, etc., are used both as general designations of forces, and also as the magnitudes of these forces when the magnitudes are not given numerically.

Magnitudes may always be regarded as essentially positive. If, in solving an equation, the value of a magnitude appears to be negative, it is to be understood, nevertheless, that the force is positive as usual, but that its sense is opposite to that assumed in stating the equation.

Direction can be expressed in **algebraic** work by the angle (α, β, etc., or 30°, 60°, etc.) which the line of action makes with, say, a horizontal line, and, by suitable convention, sense can readily be included in direction. The convention as to sense which will be used in this work will be the familiar one of trigonometry, in which the angles are measured anti-clockwise from a horizontal axis continuously from 0° to 360°, the measurement always being made to that portion of the line of action on which the sense will be away from the horizontal axis. Accordingly, for example, a force with direction 30° would be one to the right and upwards; one of 210°, to the left and downwards.

Sometimes when the line of action of a force is known but

the sense as well as magnitude unknown, it will, as above intimated, be convenient, for the algebraic determination of the sense, to permit it to be indicated by the sign of the root of the equation which determines the magnitude. The sense once determined, it should finally be associated with direction as usual.

Direction is expressed in **graphic** work by the slope of a visible line, and sense is distinguished by an arrow placed upon the line.

The point of application is expressed in **algebraic** work by coordinates (one of which it will often be convenient to make zero) and in **graphic** work by points properly plotted.

27. Space Diagram and Magnitude Diagram.—To compute graphically with forces, two distinct but intimately related diagrams are from the nature of the case necessary, viz., the **Space Diagram** and the **Magnitude Diagram.**

In the space diagram only directions and points of application are plotted, and only points of application are determined.

In the magnitude diagram only magnitudes and directions are plotted, and only magnitudes and directions determined.

The scale of the former is that of lengths, as inches, centimeters, etc., while the scale of the latter is the scale of force units, as pounds, kilograms, etc.

Forces are designated in the space diagram by small letters on each side of the line of action, and in the magnitude diagram by the corresponding capital letters at each end of the proper line. This particular style of lettering is Bow's system. It has great advantages over any other.

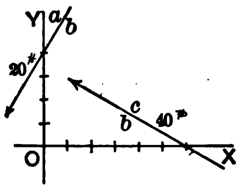
Other convenient characters may be affixed to either diagram. See Figs. 2 and 3.

28. Illustration of Scheme of Notation.—The scheme of notation enunciated in the two preceding sections may be illustrated fully by taking two forces, P_1 and P_2, their

magnitudes, directions, and points of application being assumed respectively to be 20 lbs., 240°, (0, 4), and 40 lbs., 150, (6, 0).

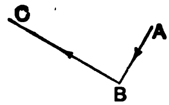
Putting them in their most condensed form algebraically, they may be written

$$P_1 = (20 \text{ lbs. } 240° \ 0, 4)$$
$$P_2 = (40 \text{ lbs. } 150° \ 6, 0).$$



Space Diagram
Scale, 1 in. = 8 ft.

FIG. 2.



Magnitude Diagram
Scale, 1 in. = 60 lbs.

FIG. 3.

Graphically the same data would be shown as in Figs. 2 and 3.

CHAPTER III.

PARALLELOGRAM OF FORCES AND ITS DERIVATIVES, THE TRIANGLE OF FORCES, THE MAGNITUDE POLYGON, AND THE STRING POLYGON.

29. Demonstration of the Parallelogram of Forces.* THEOREM.—If two concurrent forces, P and Q, both taken as acting away from their common point M, be represented in direction and magnitude by two sides of a parallelogram meeting at M, their resultant, R, is represented in all its elements by the diagonal starting at M.

PROOF.—From the corollary of § 25 it follows that the theorem will be proved if it can be shown that the sum of the moments of P and Q about all points O in the plane is equal to the moment of R about all such points O.

Let (Fig. 4) p, q, and r be the arms respectively of P, Q, and R with reference to O where O is any point whatever in their plane, and, noting the diversity in sign between the moments of P and Q, we have only to prove that $Rr = Pp - Qq$.

* The parallelogram of forces was first pointed out to the world in 1687 by Sir Isaac Newton and Varignon, probably independent of each other. The foregoing is only one of many known proofs. It is closely akin to that of Rankine, Applied Mechanics, 15th ed., p. 35. See Bowser's Analytic Mechanics, p. 24, and Weisbach's Mechanik, I, p. 165.

Many writers and teachers content themselves with the view that the whole proposition is a truth of observation and experience quite as much as the premises which have to be resorted to for its proof, and so find little or no need for mathematical proofs. But when the premises are materially more simple and more a part of common experience than the proposition itself, the resulting proof is believed to be well worth the consideration of beginners.

The point common to P and Q being M, with coordinates x and y referred to axes, OX and OY parallel respectively to

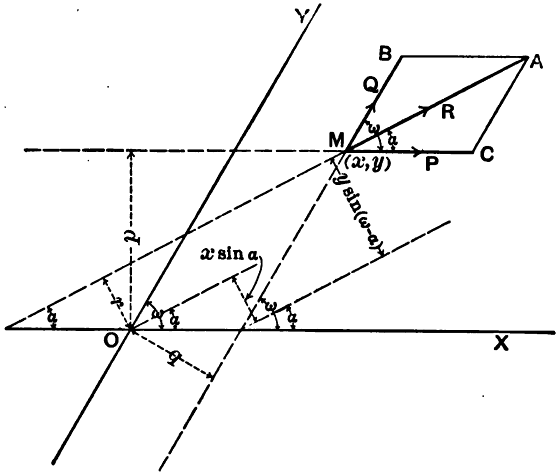


FIG. 4.

P and Q, the values of the arms can be written directly from the figure, calling the inclination of P to Q and R, ω and α respectively,

$$p = y \sin \omega,$$

$$q = x \sin \omega,$$

$$r = y \sin (\omega - \alpha) - x \sin \alpha.$$

From the figure, as a condition that R may be the diagonal of the parallelogram, by the theorem of sines

$$P = \frac{R \sin (\omega - \alpha)}{\sin \omega} \quad \text{and} \quad Q = \frac{R \sin \alpha}{\sin \omega}.$$

Substituting these five values in the equation $Rr = Pp - Qq$ as a means of testing its validity, there results

$$Ry \sin (\omega - \alpha) - Rx \sin \alpha =$$

$$\frac{R \sin (\omega - \alpha)\ y \sin \omega}{\sin \omega} - \frac{R \sin \alpha\ x \sin \omega}{\sin \omega},$$

or

$$y \sin(\omega - \alpha) - x \sin\alpha = y \sin(\omega - \alpha) - x \sin\alpha,$$

an identity,* and the theorem is proved.

Corollary I. Three forces in equilibrium must meet in a point.

Three forces may be parallel and still be in equilibrium—a fact not at variance with the preceding sentence if parallel forces be considered a limiting case with a common point at infinity.

Corollary II. With rectangular axes, and with (x, y) as any point on the line of action of any force, it appears, from the value of r, that the arm of any force referred to the origin is

$$a = y \cos\alpha - x \sin\alpha,$$

where α is the direction of the force.

30. Triangle of Forces.—Observe in the parallelogram of forces of Fig. 4 that R divides the parallelogram into two equal triangles, MAB and MAC, either of which shows fully the relations between two forces and their resultant as to magnitude and direction. Either serves the same purpose as the whole parallelogram so far as these two elements are concerned, and will be used in place of it under the name **triangle of forces.**

Observe that if the three forces are in equilibrium the arrows will follow one another around the triangle, and that in case all the arrows do not follow one another the two forces for which they do follow have the third for a resultant.

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* Observe that, whatever the location of M and the direction of the forces P and Q, the signs connecting Pp and Qq in $Rr = Pp - Qq$, and those connecting $y \sin(\omega-\alpha)$ and $x \sin(\omega-\alpha)$ in the expression for r are always identical and that the identity here noted is not dependent upon special conditions.

Observe also in the triangle of forces that one of the three forces must always be shown in a line of action not its own; in fact it will usually be found convenient to use parallels to the actual lines of action of all the forces. This gives rise to the magnitude diagram of § 27.

From the triangle of forces it follows that if α be the inclination of a force P to a given line, $P \cos \alpha$ and $P \sin \alpha$ measure the components of P parallel and normal to that line respectively (cf. Cor., § 17).

31. Polygon of Forces. Location of Resultant of Inclined Forces. Magnitude Polygon.—The resultant of any set of forces can be determined in magnitude and direction by forming a polygon with its sides proportional to the magnitudes of the given forces, and parallel to them, with arrows following one another. The magnitude and direction of the resultant will be shown by the closing line of the polygon, the proper sense being away from the starting-point.

This follows immediately from the triangle of forces, as will be seen from Fig. 5, where the process of finding the magnitude and direction of the resultant of the four forces *ab*, *bc*, *cd*, and *de* is shown in detail.

AB combined with BC by the triangle of forces, Fig. 5*b*, leads to their resultant AC, which is in position to be combined at once with CD to yield AD—the desired resultant of AB, BC, and CD. This process can obviously be continued indefinitely. The partial resultants are seen to be actually superfluous in the construction, and the polygon $ABCDE$ might have been built up without them. This polygon is commonly called the **polygon of forces**, but the more distinctive name, the **magnitude polygon** will be preferred in this book.

Observe that while the magnitude polygon does not locate the points of application of AC, AD, or AE, it still furnishes important aid in finding them. We need only to observe that,

in accordance with the parallelogram of forces, the point of application of AC is at the intersection of ab and bc. Then ac

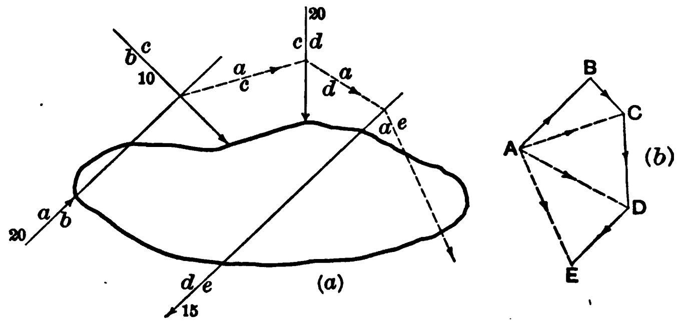

FIG. 5.

can be drawn, and where it cuts cd is a point of ad. Drawing ad, where it cuts de is a point of ae, and so on for any number of forces.

A method applicable to parallel forces will be developed presently.

Corollary. If the last point of the magnitude polygon coincides with the first point, i.e. if the polygon closes, the resultant is zero and the set of forces is equivalent to a couple or is in equilibrium, according to the relative positions of the given forces.

32. String Polygon.—As has been seen, the magnitude polygon takes into account only magnitudes and directions of forces. The next step is to establish the general method by which points of application are taken into account and determined. This method involves a construction in the space diagram in close relation with the magnitude polygon, which may be developed as follows.

With any given set of coplanar forces, assume the addition of any force whatever, its magnitude, direction, and point of application being chosen entirely at random or to suit con-

venience. Find, by aid of the magnitude polygon, the magnitude and direction of the second force that would have to be added to the set to close the magnitude polygon for the given set *plus* the assumed force. This second force would be the resultant of the new set, and if properly located in accordance with the method of § 31, would be a force which, together with the assumed force, would make a set of two forces which would be equivalent to or balance the given set according to the senses ascribed to them.

The senses of the assumed force and its mate will be confluent in the magnitude polygon, and if they are also confluent with the rest of the set, the two would balance or equilibrate the given set; if they are not so confluent, they would be two forces together equivalent to the given set.

Thus two forces can always be determined, which will be equivalent to any given force or to any given set of coplanar forces, no matter how complicated, and they will furnish the desired means for taking account of points of application.

It may be observed that these two forces must always fall within one of five cases which may be stated and interpreted as follows:

(*a*) *The two forces may be inclined to each other and either equal or unequal in magnitude.* By the parallelogram of forces their resultant must pass through their point of intersection. But their resultant and that of the given set are identical. Therefore their intersection is a point on the line of action of the resultant of the given set. Their non-parallelism would show that there must be such a resultant.

Observe that this method furnishes a means for locating the resultant of a set of parallel forces.

(*a'*) *The two forces may be parallel, unequal, and opposite,* proving the given set to be reducible to a single force parallel to the two forces and acting at a point determinate only by re-

placing the two forces in their turn by two others equivalent to them and inclined to each other as in (a). This may be regarded as a special case of (a). This case can always be avoided in practice by judicious selection of the direction of the assumed force.

(a'') *The two forces may be coincident, unequal, and opposite,* proving the given set to be reducible to a single force, itself coincident with the two coincident forces. This like (a') may be regarded as a special case of (a) and like (a') can always be avoided by proper selection of the assumed force.

(b) *The two forces may be parallel, equal, and opposite,* forming a couple, and proving the given set to be reducible to a couple.

(c) *The two forces may be coincident, equal, and opposite,* that is in equilibrium, and proving the given set to be in equilibrium.

The cases (a), (a'), and (a'') are evidently cases in which the magnitude polygon is not closed, and (b) and (c) are evidently cases in which that polygon is closed.

In Figs. 6a, 6b, 6c are illustrated the important cases (a), (b), and (c) respectively. OA and oa are assumed outright and EO and eo or FO and fo determined just as AE and ae were determined in Fig. 5. In Fig. 6a it appears that the given set amounts to a single force AE; in Fig. 6b to an anti-clockwise (negative) couple of value $OA \times p$; and in Fig. 6c that they are in equilibrium, A and F coinciding and also ao and af. In Fig. 6a the case (a') would have resulted if O had been taken anywhere on AE; oa and oe would then of course be parallel, becoming coincident, case (a''), if oa should chance to contain the point common to ab and ae.

Observe that, in Fig. 6a, oa and oe, and in Figs. 6b and 6c, oa and of, with the senses there shown, are pairs of forces *equivalent* to the given set.

The forces *oa*, *ob*, *oc*, *od*, etc., form a polygon consisting of the lines of action of an arbitrarily chosen force *oa*, and the

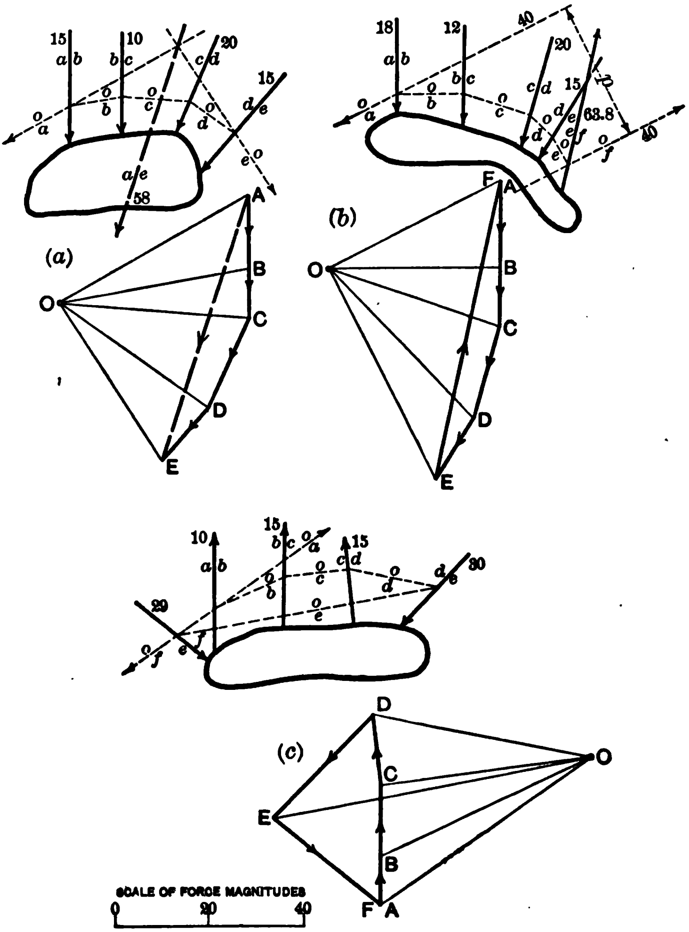

FIG. 6.

resultants respectively of *oa* and *ab*, of *oa*, *ab*, and *bc*, etc., through the series of given forces. This polygon is called the **string polygon**, because it bears a certain relation to the form

which a string would assume if secured at the ends and subjected to the given forces.*

If the two added forces equivalent to or balancing the given set coincide, the string polygon forms a closed figure and is described as **closed**.

Obviously, then, if the string polygon closes, the given set of forces cannot be reducible to a couple.

Observe that, in Fig. 6*a*, *ae* was located where *ao* equal and opposite to *oa* would have given rise to a closed string polygon.

33. Additional Remarks on the String Polygon.—The sides of the string polygon are called strings. The added force, *OA*, and the partial resultants, *OB*, *OC*, *OD*, etc., form a set of lines radiating from a common point (called the **pole**) and are hence known as **rays**. The set of rays and the magnitude polygon constitute the whole of the magnitude diagram.

To construct a closed string polygon, one need only to see that the strings form a closed polygon having an apex on the line of action of each and every force of the system (and nowhere else),—each apex to be formed by the pair of strings which represent components of the force on whose line of action the apex appears.

Any one string is, of course, a closing line of a closed string polygon, but any string which it may be convenient to draw last will commonly be spoken of as the closing line.

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* Terms more strictly analogous to magnitude polygon would be point-of-application polygon or location polygon but they are not in use. Funicular polygon and equilibrium polygon are terms in common use as well as string polygon, the former being merely the Latin equivalent to string polygon, and the latter open to the objection that it might with equal appropriateness be applied to the magnitude polygon.

CHAPTER IV.

ALGEBRAIC AND GRAPHIC STATEMENTS OF THE CONDITIONS OF EQUILIBRIUM WITH APPLICATIONS.

34. **Statical Problems.**—Problems in statics deal with bodies which may be conceived to be at rest under the action of a set of forces, some of which are not fully known, i.e., not known as to all their elements. The solution of such problems consists of finding what must be the value of each of the unknown elements. This at once suggests the use of algebra, and the first and principal task is to find how to write equations which will truly represent the conditions—that is, which will be true *only if equilibrium is established.* The forces can be represented in such equations only by expressions for their elements. Such elements as are unknown need only be expressed as unknowns, to be evaluated as usual by the ordinary process of solving the equations. To write such equations, one need only to consider carefully what must be true that a body may be at rest, i.e., what the conditions really are which are to be expressed.

35. **Conditions of Equilibrium.**—As has been pointed out (§ 13), conditions of rest are always conditions of equilibrium, and to arrive at the conditions of equilibrium it is only necessary to consider the conditions of rest.

In order that a body may be at rest, two things must be true of the forces acting on it, viz.:

(*a*) The total tendency of some of them to produce trans-

lation in one direction must be met by an equal total tendency on the part of the others in the opposite direction.

(b) The total tendency of any of them to produce rotation in one direction must be met by an equal total tendency on the part of the others in the opposite direction.

Accordingly **a set of forces will be in equilibrium if they can result in**

(A) **Neither translation**

(B) **Nor rotation.**

Observe that this is the same thing as saying that there must be for a resultant neither (A) a single force nor (B) a couple.

It remains to establish the algebraic equivalent of (A) and (B), and then the graphic equivalent. It will be best for the present to confine attention to coplanar forces.

36. Algebraic Statement of (A) and (B) for Coplanar Forces.—Since, in a given problem, the directions of the forces may be very various, it is necessary in algebraic work to subdivide translation in general into two component translations parallel to a set of axes, preferably rectangular. For coplanar forces a pair of such axes will of course suffice. They will be most conveniently taken as horizontal and vertical and will be referred to as the axis of X and the axis of Y respectively, according to the usual convention.

(A) will then be subdivided into two partial statements which will be distinguished by the proper subscripts, and the conditions can now be written

(A_x) No translation to the right or left;

(A_y) No translation up or down;

(B) No rotation.

But the tendency of a force to produce translation to the right or left is simply the horizontal or X-component of the force and can be expressed algebraically $P_1 \cos \alpha_1$, $P_2 \cos \alpha_2$, etc.

And the tendency of a force to produce translation up or down is simply the vertical or Y-component of the force, and is expressed algebraically $P_1 \sin \alpha_1$, $P_2 \sin \alpha_2$, etc.

In order to express (B) algebraically, it will be convenient to use the principle of § 20 and consider the given set replaced by an equivalent set consisting of a series of concurrent forces and a series of couples. The point of concurrence may be chosen to suit convenience and be called m. The couples need to be considered further only in connection with rotation. The moment of each may be regarded as its contribution towards rotation, and the values of these moments are $P_1 a_1$, $P_2 a_2$, $P_3 a_3$, etc. If now m be taken as the origin of coordinates, and (x_1, y_1), etc., be taken as coordinates of the points of application of the forces, it will be convenient to insert the equivalents for the a's, and the moments will appear in the form $(P_1 y_1 \cos \alpha_1 - P_1 x_1 \sin \alpha_1)$, $(P_2 y_2 \cos \alpha_2 - P_2 x_2 \sin \alpha_2)$, etc., which may be looked upon indifferently as the sum of the moments of the components or as direct substitution of values of the a's. Cf. Cor. II, § 29.

(A) and (B) become, accordingly, expressed as equations:

$$(A'') \begin{cases} P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \dots + P_n \cos \alpha_n = 0. \quad . \quad A''_x) \\ P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + \dots + P_n \sin \alpha_n = 0. \quad . \quad A''_y) \end{cases}$$

$$(B'') \quad P_1(y_1 \cos \alpha_1 - x_1 \sin \alpha_1) + \dots \\ + P_n(y_n \cos \alpha_n - x_n \sin \alpha_n) = 0,$$

or, more briefly,

$$(A') \qquad \begin{cases} \Sigma P \cos \alpha = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad A'_x) \\ \Sigma P \sin \alpha = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad A'_y) \end{cases}$$

$$(B') \qquad \Sigma P(y \cos \alpha - x \sin \alpha) = 0,$$

or, representing all horizontal and vertical components by X and Y respectively, these three equations take the form $\Sigma X = 0$, $\Sigma Y = 0$, and $\Sigma(Xy - Yx) = 0$, or most simply of all, using M for moments,

$$(A) \qquad \begin{cases} \Sigma X = 0. \quad \ldots \ldots \quad A_x) \\ \Sigma Y = 0. \quad \ldots \ldots \quad A_y) \end{cases}$$

and

$$(B) \qquad \Sigma M = 0. \quad \ldots \ldots \quad B)$$

This last is the usual form for the algebraic statement of conditions of equilibrium. These equations form an equipment for solving statical problems algebraically.

Two other ways of insuring the satisfaction of (A) and (B) algebraically can be deduced, one of much practical importance, the other of little. These will receive attention in due time (§§ 40, 41, 43), and will be shown to be mere transformations of the equations above stated.

Note that the satisfaction of (A) precludes translation, and that the satisfaction of (B) precludes rotation.

Show that the satisfaction of any one or any two of the A_x, A_y, and B equations will not insure equilibrium.

Show that in case of concurrent forces (A) alone suffices, and gives the only two independent equations bearing on the case, i.e., that (B) will always hold in this case if (A) does.

When then are both (A) and (B) always needed?

Must three forces in equilibrium be concurrent? Why? How about three parallel forces? How about more than three forces? Cf. Cor. I, § 29.

37. Graphic Interpretation of (A) and (B).—That there may be no translation from a given set of forces it is evident that their resultant must be zero, and by § 31 it is seen that

this can be only when the magnitude polygon for the set of forces closes. We can state then

(A) **A magnitude polygon must close.**

That there may be no rotation, the given set of forces must not be reducible to a couple, and, as has been shown in § 32, this condition will be fulfilled if a string polygon closes for the set of forces. We can state accordingly

(B) **A string polygon must close.**

The closed magnitude polygon precludes translation, and the closed string polygon precludes rotation. These two polygons constitute the full equipment for the solution of statical problems graphically. It remains only to get practice in using them.

Here again, of course, for concurrent forces (A) alone suffices, and its satisfaction carries with it the certainty of satisfaction of (B). Hence with concurrent forces (B) is superfluous.

38. **Exercises in the Composition of Forces.**—The equipment for solving statical problems is now complete, except the transformation of the A_x, A_y, and B equations referred to in § 36, and developed in §§ 40, 41. Everything is, however, entirely ready for practice in the composition of forces, and practice in the use of what has already been developed will throw much light on what is to follow.

Below are given explanations of the algebraic and graphic methods of procedure in a general case. In special cases more or less of both processes become superfluous and can be omitted after a little practice. The plan for arrangement of work and for retention of memoranda at all steps is recommended as, in the long run, an economizer of time, both because it di-

minishes the risk of numerical error, and because it aids in the location of such error when once committed.

Let it be required to determine the equilibrant (and resultant) of any set (say five in number) of non-parallel, non-concurrent coplanar forces.

Graphic Solution. (Figs. 7*a*, 7*b*. Cf. Pl. I.) Letter the five forces in the given set, preferably in the order of their occurrence from left to right, *ab*, . . . , *de*. The required equilibrant will be *ea* and the resultant *ae*.

Construct (Fig. 7*b*) the magnitude polygon (§ 31) *ABCDE* for the given forces, the sides having lengths proportional to the magnitudes of the forces to which they are parallel and so arranged that the arrows will follow one another. Evidently all that is needed to close this polygon now is to fill in the side *EA*. *EA* determines the magnitude and direction of the equilibrant, and *AE* those of the resultant.

Construct the string polygon (Fig. 7*a*). The question now is how to locate *ea* so as to preclude rotation, i.e., how to locate *ea* so that the string polygon will close when this force is added to the set.

Now *ea* must act at the intersection of strings *oe* and *ao*, and to close the string polygon *ao* must coincide with *oa*. The position of *ao* thus being known and *oe* having already been located in the construction of the string polygon up to this point, it only remains to draw *ea* through their point of intersection. The string polygon is then seen to be closed, and any convenient point on *ae* may be taken as the required point of application.*

Algebraic Solution. (Cf. Pl. I.) Let the unknown magni-

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* The short dash-lines in Fig. 7*a* show *ea* in other positions than the correct one and the positions which *ao* would in those cases take, leaving the string polygon unclosed. Such forces *ea* and *ao* are there distinguished by subscripts.

tude, direction, and point of application be designated E, α, and (x, y) respectively. Write the equations for horizontal

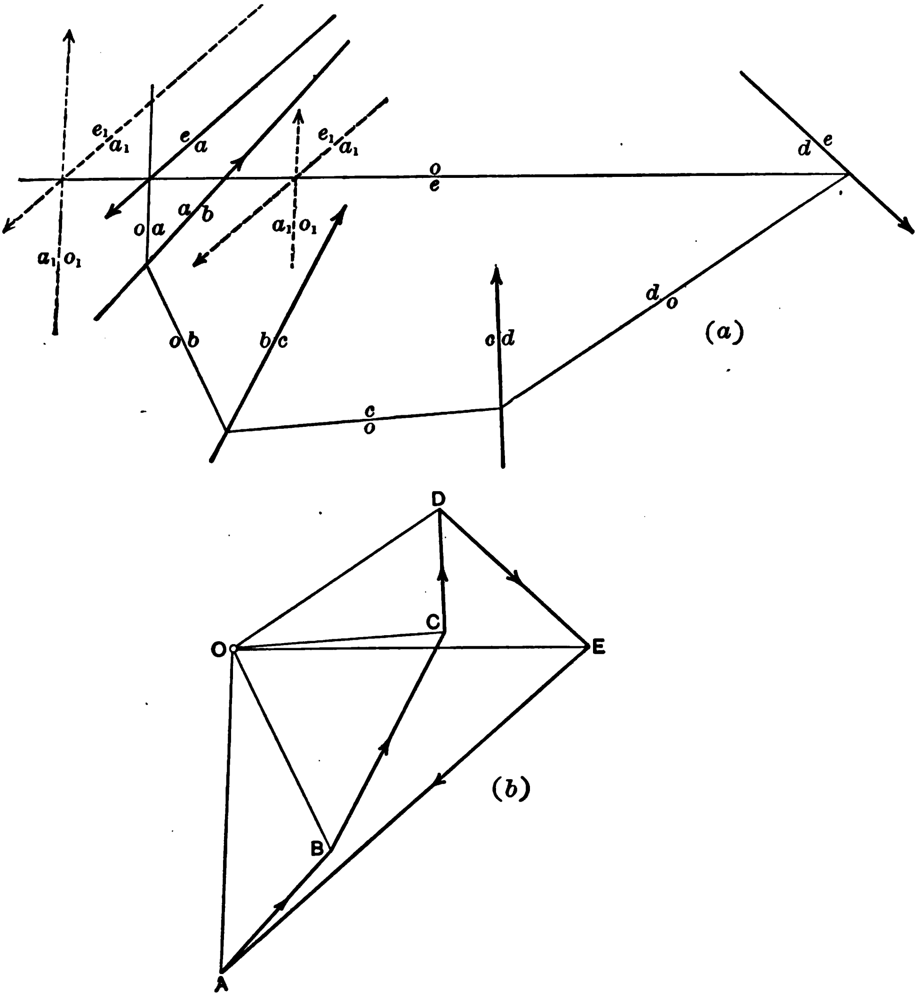

FIG. 7.

and vertical components—the A_x and A_y equations—the second considerably below the first.

Insert in small figures, as memoranda, the values of the

trigonometric functions just above each, taking care, by a moment's inspection, not to interchange the values of the sines and cosines.

Perform the multiplications indicated and insert the products in small figures as memoranda below each term.

Proceed with the reduction as shown in Pl. I, and determine the values of E and α, observing that the α found will be that for the equilibrant and needs only to be changed by 180° to give the α for the resultant. The quadrant indicated for α is the first, second, third, or fourth according as $E \sin \alpha$ and $E \cos \alpha$ are both plus, the former plus and the latter minus, both minus, or the former minus and the latter plus.

Write the equation of moments—the B equation—in extended form, inserting the values of the given coordinates at once, striking out all terms in which a zero coordinate appears.

Write as memoranda above each remaining $P \cos \alpha$ and $P \sin \alpha$ the value found for it in reducing the two preceding equations.

Perform the multiplications indicated and show products under each term as memoranda, and solve, assuming the value for either x or y as may be more convenient.

Specifications for Exercises 3–7. In each exercise the object is to work out both graphically and algebraically, the equilibrant and resultant of the sets of coplanar forces there given. The results should be recorded side by side conspicuously for comparison. Work will be arranged in general as in Pl. I, and each exercise should be undertaken as a general case and so carried to completion.

Magnitudes are expressed in pounds and lengths in feet.

Angles will best be laid off by plotting from a table of natural tangents, and in favorable cases by the use of triangles and T-square. Small protractors will sometimes be found useful for checking.

For scales, 1 in. = 20 lbs. and 1 in. = 4 or 5 feet are recommended.

Exercise 3. Non-concurrent, non-parallel forces: (20 105° 5, 0), (25 263° 11, 0), (30 98° 15, 0), (10 80° 17, 0), (15 280° 2, 0).

Exercise 4. Parallel forces uniform in sense; five vertically upward forces of 10, 20, 30, 40, and 50 lbs. respectively, with successive intervals between them of 3, 2, 6 and 5 ft.

Exercise 5. Parallel forces varying in sense: the forces of the preceding exercise but with the 10-lb. and 30-lb. forces reversed in sense.

Exercise 6. Concurrent forces: five forces of 10, 20, 30, 40, and 50 lbs. respectively, with directions of 30°, 60°, 135°, 220°, and 320° all applied at (0, 0).

Exercise 7. Three non-concurrent forces whose magnitude polygon closes: (50 330° — 6, 0), (50 210° 0, 2), and (50 90° 8, 0).

NOTE.—This is a case similar to that of Fig. 6*b*, § 32. The results can be stated in foot-pounds with the proper algebraic sign to indicate sense, say + for clockwise and — for anti-clockwise rotation. The source of graphical results should be made clear by use of dimension lines, including scaled numerical results and the expression of the necessary multiplication. There is no reason why *OA* may not be taken of some round magnitude, say 20, 30, etc., and it will be advantageous in this case.

39. Generalization of the Three Classes of Resultants.—It may be observed in passing that the three classes of resultants to one of which any set of coplanar forces may be reduced can be expressed in generalized form in the notation of this book, as follows:

Class 1. (>0, 0° to 360°, unrestricted), i.e., a single force. In this case the magnitude polygon fails to close, and the string polygon may or may not close.

Class 2. (0, indeterminate, at infinity), i.e., a couple. Here the magnitude polygon closes, but the string polygon does not close.

Class 3. (0, indeterminate, indeterminate), i.e., a set in equilibrium. The magnitude polygon and the string polygon both close.

Examples of the first two of these classes are contained in the data for Exercises 3–7. Inserting the equilibrant there required, the class then exemplified is in each case the last one, of course.

40. Establishment of Equilibrium by Use of Moments Alone.—While the algebraic statement of equilibrium of § 36 is, of course, of universal validity in cases of coplanar forces, there is an alternative algebraic statement which in many cases

proves much more convenient in use. This statement takes into account moments only, and amounts simply to three equations of moments with certain limitations laid upon the choice of the centers of moments. This alternative is deduced as follows.

To establish equilibrium, it must be made certain that the given set of forces will cause neither translation nor rotation, i.e., that they amount to neither a single force nor a couple. It has been shown that there can be no couple if $\Sigma M = 0$ for any point whatever. It remains to be shown how moments can be used to prove the non-existence of a single force resultant as well.

Suppose that for a given center of moments $\Sigma M = 0$ for a given set of forces. This can arise from only one of two causes: either the set of forces is in equilibrium, and would have no moment about any point, or the center of moments lies upon the resultant of the set. If, then, it can be shown that the center of moments does not lie on the resultant of the set, the set must be in equilibrium.

Any one or any two points chosen at random might both chance to lie on the line of action of a resultant. In such a case ΣM would be zero in spite of the resultant having a magnitude greater than zero, and if investigation went no further translation might exist undetected. If, however, **three points not in the same straight line** be chosen, one of the points at least will not lie on such line of action. If $\Sigma M = 0$ for **each** of these points, $\Sigma M = 0$ for at least one point not on the line of action of the resultant, and equilibrium is a certainty. Hence it can be stated that

If, for a set of forces, ΣM is zero for three points not in the same straight line, equilibrium is assured.

It should be carefully observed that the three equations of moments implied in the preceding sentence are merely the A_x,

A_y, and B equations themselves, though transformed * in such a way as to be more convenient for use in certain problems. As mere variants or alternative forms of the equations of equilibrium first deduced they can solve no problems which the former could not be made to solve, and any general conclusions drawn from inspection of either of the two sets of equations could be drawn also from inspection of the other. The A_x, A_y, and B equations, as they are of simpler algebraic form, are more convenient for such inspection, but, nevertheless, they are a group of equations algebraically identical with the three moment equations.

41. Determination of Magnitudes by Single Moment Equations Alone.—Complete determination of equilibrium requires in general three equations, A_x, A_y, and B, or an equivalent, such as the three moment equations of § 40. Still it will be worth while to see under what circumstances, if at all, a single equation of moments will determine the magnitude of any one of three or fewer forces whose lines of action are known.

Call this unknown force Q, and the others R and S. Then, since Q is to be the equilibrant of all the rest of the set, its line of action must be that of the resultant of the set.

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* The nature of this transformation appears upon writing one of these moment equations in its general form as follows.

If the points of application of the forces be a series of (x, y)'s, and any one of the centers have the coordinates (x_a, y_a) with reference to the same axes, the moment equation will take the form

$$\Sigma[X(y - y_a) - Y(x - x_a)] = 0;$$

or, noting that x_a and y_a are constants and all the rest of the letters variables,

$$\Sigma(Xy - Yx) - y_a \Sigma X + x_a \Sigma Y = 0,$$

which is obviously B minus A_x, multiplied through by a constant and plus A_y multiplied through by another constant. One or both of these constants might, of course, be zero. If the three centers be taken at the origin and elsewhere on each of the axes of x and y respectively, the three moment equations become simply B, and $B - A_x \times$ a constant and $B + A_y \times$ a constant.

Realizing this it is easy to select a center which certainly does not lie on the line of action of the resultant of the set. R and S, as well as Q, will in general produce moments about the point, and, appearing in the equation, will prevent Q from being the only unknown in it. If, however, the point can be selected so that it will be off the line of Q, and at the same time cause the arms of R and S, or of S if S is the only other unknown, to be zero, determination of Q by one equation is possible. Such a point is to be found at the intersection of R and S, or anywhere on S if S is the only other unknown. Cases in which this cannot be done arise only when there are more than three unknowns, or when the lines of action of three are parallel or concurrent.

An equation thus derived of course does not in general establish equilibrium, for the establishment of equilibrium requires the determination of two or three elements, and for this more than one equation is always necessary; but it does determine one magnitude to which the other forces must inevitably conform in order to produce equilibrium. To get more, simply repeat the process. In general, then,

To find the magnitude of a force whose line of action is given, solve an equation of moments for a center on the line of action of the other force whose magnitude is unknown; or, if there are two others, for a center at their intersection. Repeat the process for the other one or two forces.

This method is of great use whenever the magnitude is sought for a force with a given line of action. Such problems are very common in practice. In such cases the method is often useful for determining senses of forces by mere inspection without any calculation.

Notice that solving Case 4 in this way is simply utilizing

the general principles of the preceding section, choosing the centers so as to avoid simultaneous equations.

The amount of it is that if there is equilibrium, $\Sigma M = 0$ for any center, but the converse is not true. Hence the need of more than one moment equation.

Suppose the condition requiring the center to be off the line of action of Q were violated, how would the equation be affected?

42. Convention as to Algebraic Signs in Moment Equations.—In using A_x, A_y, and B it has been convenient to consider magnitudes of forces as always positive and to associate sense exclusively with slope. In using the method of moments it will be more convenient to call arms universally positive, and give magnitudes varying signs: $+$ if for a given center they cause rotations in one sense, and $-$ if the opposite. Throughout these pages clockwise rotations will be called positive, and anti-clockwise negative.

To apply the method of moments, select the center, write the equation calling all moments positive that are not known to be negative. The sign found with the result will, **in connection with the center for the equation,** determine the sense of the force. The same force may be positive for one center and negative for another, hence the locations of the centers should be recorded near the work.

43. Six Methods of Stating the Conditions of Equilibrium.—In addition to the statements of the conditions of equilibrium already given, there is a third from the algebraic and two more from the graphic point of view. The six ways are here collected and stated for completeness, and for reference:

*Algebraic.**	*Graphic.*
Equilibrium will exist if	Equilibrium will exist if
(1) **The sum of the moments is zero for each of three points not in the same straight line;**	(1) A string polygon closes for each of three poles not in the same straight line;
or if	or if
(2) The sum of the moments is zero for each of two points, and the sum of the components is zero along a line not perpendicular to the one joining these two points;	(2) Two string polygons close with the same pole;
or if	or if
(3) **The sum of the moments is zero for one point and the sum of the components is zero for both of any two directions.**	(3) **One string polygon and one magnitude polygon close.**

$\Sigma M_A = 0$
$\Sigma M_B = 0$
$\Sigma X = 0$

The statements shown in bold face are the ones already developed at length and are the only ones of practical use. The critical reader, however, will assure himself of the correctness of the three others.

* The first of these statements was shown in § 40 to be an algebraic identity with the third, and deducible directly from it, and the same might be proved of the second in a similar manner.

CHAPTER V.

SCOPE OF PURE STATICS.

44. General Survey of the Scope of Pure Statics.—The term **Pure Statics** is to be understood to mean the perfectly general science, in which problems can have no light thrown on them by considerations of size, shape, and elasticity of bodies. Its only resources are the conditions of equilibrium of rigid bodies.

The first algebraic statement of these conditions which has been elaborated is of such a form as specially to invite thought as to the scope of the subject—as to how many and what coplanar statical problems are capable of solution.

First may be considered

(*a*) *Non-concurrent Forces*. There are only three fundamental equations (§ 43), A_x, A_y, and B. If, then, in any problem there are more than three forces of which elements are not known, there are more than three unknown quantities and the problem of establishing equilibrium is indeterminate. Hence all forces but three (at most) must be fully known. But three forces involve nine elements; hence, unless six of these elements are known the problem is still indeterminate.

These are *a priori* considerations regardless of the structure of these particular equations. Taking this structure into account, the field becomes still more circumscribed. For example, since only one of the equations involves points of

application, it is clear that there must be not more than one such point unknown. Even then the point is, strictly speaking, indeterminate, for the point appears as two unknown quantities, x and y. For the purposes of statics, however, any pair of values for x and y that will satisfy the equation will meet all requirements, and this actual indetermination is not a source of difficulty. In other words, B becomes simply the equation of the line of action of the force in terms of x and y, on which an indefinite number of points of application may be selected.

Nine elements can be divided into groups of six known and three unknown in $9 \times 8 \times 7 \div 3 \times 2 = 84$ different ways, which reduce to twenty separate cases at once (see Appendix for a complete statement of them), and of these, fifteen are more or less indeterminate, owing to the peculiar limitations of the equations above pointed out. Moreover, only three of the twenty cases are important. They are the following:

α. When the three unknown elements all pertain to one force, i.e., one force is wholly wanting. This is the problem of finding the resultant and the equilibrant of a given set,—Composition of Forces.

β. When the three unknown elements pertain to two forces and the magnitude of one and the magnitude and slope of the other are wanting.

γ. When the three elements pertain to three forces, and three magnitudes are wanting.

There are still to be considered

(*b*) *Concurrent* (*including Parallel*) *Forces.* Here there are only two fundamental equations, A_x and A_y,* one of

* The B equation becomes simply a mathematical identity with A_x and A_y in the case of concurrent forces, for the x's and y's in $\Sigma(Xy - Yx) = 0$ are then each constant, say x_a and y_a. The equation is then $y_a\Sigma X - x_a\Sigma Y = 0$, which is $A_x \times y_a - A_y \times x_a$.

which must be replaced by B if the forces are parallel. They can determine only two unknown quantities. If there are more than two forces of which elements are unknown, there are more than two quantities unknown and the problem of establishing equilibrium is indeterminate. Hence all forces but two (at most) in a set of concurrent or parallel forces must be fully known. But two forces involve six elements which may be unknown, and unless two of these elements are known, the problem is indeterminate. Six elements may be divided into groups of four known and two unknown in $6 \times 5 \div 2 = 15$ different ways, which reduce to nine separate problems in the equilibrium of concurrent or parallel forces (see Appendix). Noting that composition of such forces is included in the general problem of composition of forces (α) as a special case, only one of the rest of the nine cases is important, viz.:

δ. When the two elements pertain to two forces, and the magnitudes of the two forces are wanting,—Resolution of Forces.

45. The Four Cases.—Only four cases capable of solution by pure statics are of sufficient importance to require special study. These have been pointed out in § 44, and will here be restated and numbered for future reference.

Case 1. Magnitude, direction, and point of application of one force required,—Composition of Forces.

Case 2. Magnitudes of two forces required,—Resolution of Forces.

This case will be divided into two sub-cases, 2*a* and 2*b*. In 2*a* the two forces are non-parallel; in 2*b*, parallel.

Case 3. Magnitude of one force and magnitude and direction of another required.

Case 4. Magnitudes of three forces required.*

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* As will be seen (§ 50a), Case 4 may be regarded as a mere combination of Cases 3 and 2*a*, and Case 3 as combination of Cases 4 and 1.

This case is sometimes spoken of as Resolution in Three Directions.

Case 4 is statically determinate only when the three forces are neither concurrent nor parallel.

All these four cases or any of the few others which are capable of solution can, of course, be solved by applying A_x, A_y, and B—in some cases more conveniently in the form of the moment equations of § 40 or § 41—or by closing the magnitude and string polygons.

See Plates I–V respectively for a full treatment of these four cases. These plates are explained in the following chapter.

CHAPTER VI.

SOLUTION OF STATICAL PROBLEMS, WITH SPECIAL REFERENCE TO THE FOUR MOST IMPORTANT CASES.

46. The Solution of Statical Problems.—In general the first thing to be done in undertaking a problem in statics is to draw a sketch showing all that is known of all the forces,—their magnitudes, either as knowns or unknowns, being inscribed near their lines of action. This operation is of fundamental importance and is the step of greatest practical difficulty. It is sometimes called "showing the forces." It consists really of accurately analyzing the situation and deciding just what forces are acting and just what is known of each. This once done, and the problem seen to be determinate, it is a matter of mere routine to proceed with the solution.

The mastery of these solutions is what study of this chapter is expected to bring about. The ability to show the forces will require a longer time to cultivate, and its importance may well be kept in mind.

The methods of solution for the four important cases will now be taken up. It is well to note at the outset that the whole equipment available consists of the magnitude polygon and the string polygon for graphic solutions, and of the A_x, A_y, and B equations, or their alternative, the three moment equations of § 40 for algebraic solutions. The task is now simply to become familiar with the manipulations by which this equipment can be made to fit the peculiarities of each case, and to yield most easily the desired results.

In all cases, it should be noted, it will be found most convenient **in graphical work** to letter the known forces in the order of their occurrence, going from left to right around the body on which they act, leaving the unknowns to be lettered consecutively after the knowns have all been provided for. The first letter will naturally be *a*, and the last letter of the last unknown may properly be *a*. Thus will be avoided double letters for apices of the magnitude diagram.

Furthermore **in algebraic work** senses of forces given only in line of action will be assumed to be such as to make their directions less than 180°, in writing the A_x and A_y equations, and such as to give a positive moment in writing the non-simultaneous moment equations required for some of these cases. If the sense so assumed is the correct one, the fact will appear by the magnitudes proving to be plus quantities. If the magnitudes prove negative, of course the correct sense is the reverse of that assumed.

After writing an equation of moments it will be well to check the lengths of all arms, whether given or not, by scaling them from a drawing, if one be at hand. Check also so far as possible the determinations of senses by inspection of the drawing.

47. Solution of Case 1.—Case 1 is mentioned here merely for completeness, and the account of its solution given in § 38, and presumably familiar to the reader by this time, need not be reprinted here.

48. Solution of Case 2a. (Cf. Pl. II*a*.)—Required the magnitudes (and senses) of two non-parallel forces in a set in equilibrium.

Algebraic Solution. Let P and Q be the unknown magnitudes, and assume trial senses as per the next to the last paragraph of § 46.

Solve, for P and Q, either (*a*) the A_x and A_y equations

(avoiding simultaneity, if desired, by taking one axis of reference along one of the unknowns), or (*b*) their variants, the two moment equations of § 41. The assumption as to sense made at the outset is correct or must be reversed for either force according as its magnitude comes out plus or minus.

Graphic Solution (Fig. 8 and Pl. II*a*). Close the magnitude polygon (Fig. 8*b*). Suppose the first known of the given set be *ab* and the two unknowns *de* and *ea*. The magnitude polygon can be completed by familiar means from the apex A to D inclusive. Then DE drawn parallel to *de* and AE parallel to *ae* will locate the missing apex E, and the required magnitudes and senses will be DE and EA.

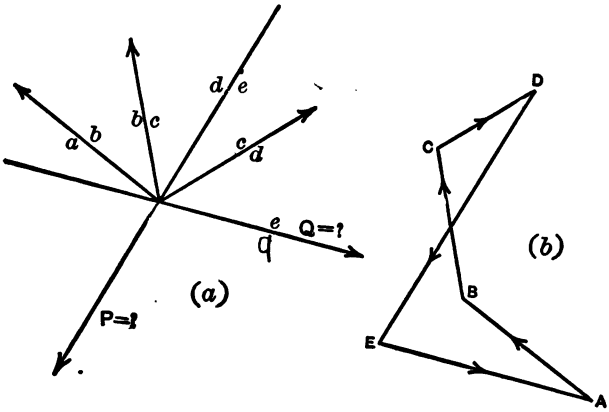

FIG. 8.

The string polygon is superfluous. It is bound to close if the magnitude polygon is closed.

NOTE.—Case 2*a* nearly always occurs in practice for a set of concurrent forces, but it is capable of solution as well for a non-concurrent set if the resultant of all the knowns passes through the point of intersection of the two unknowns.

48a. Solution of Case 2b. (Cf. Pl. II*b*.)—Required the magnitudes (and senses) of two parallel forces in a set in equilibrium.

Algebraic Solution. Let P and Q be the unknown magnitudes and assume trial senses as per the next to the last paragraph of § 46.

As P and Q are parallel, the X-axis may well be taken at right angles to them both, making their directions 90°.

Write the A_x and A_y equations as in Case 2*a*. Noting that the A_x equation vanishes, its place is made good by the B equation (retaining the same senses as in the preceding equations) written as in Case 1. Solving the simultaneous equations, P and Q are known, and the assumed senses are correct or to be reversed according as results come out plus or minus.

Or, *better*, solve by moments alone by the method of § 41, taking a center on the line of action of P to find Q, and on

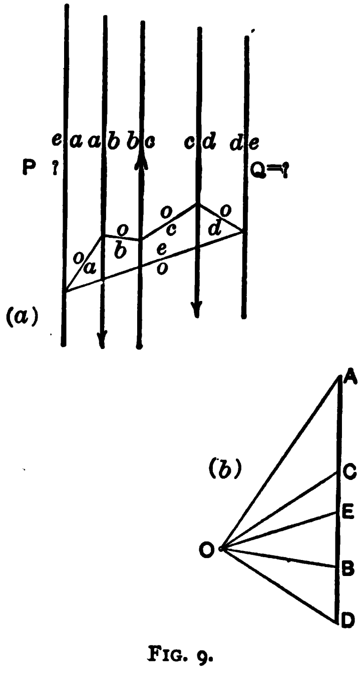

FIG. 9.

the line of action of Q to find P. The assumed senses are correct or to be reversed according as results come out plus or minus. Check results by seeing whether A_y is satisfied finally.

Graphic Solution (Fig. 9 and Pl. II*b*). Let the knowns be lettered *ab* . . . *cd*, and the unknowns accordingly *de* and *ea*.

Construct the magnitude polygon from A to D, inclusive (Fig. 9*b*). Assume a convenient *oa* and construct the string polygon to *od*, inclusive (Fig. 9*a*). In order that *ao* may coincide with *oa* and close the string polygon, *oe*, the missing string, must intersect *ea* where *oa* does, and *oe* must start from the intersection of *od* and *de*. Hence *oe* is located, and the string polygon is closed.

E must then be on a ray from O parallel to *oe*, and on a line from A or D parallel to *ae* or *de*. E is therefore located, the magnitude polygon is closed, and the required magnitudes and senses will be DE and EA.

NOTE.—This case almost always occurs in practice for a set of parallel forces, but it is capable of solution as well for any set provided that the resultant of the knowns is parallel to the two unknowns.

49. Solution of Case 3. (Cf. Pl. III.)—Required the magnitude and direction, and the magnitude (and sense) respectively, of two forces in a set in equilibrium.

Algebraic Solution. Let P and Q be the two unknown magnitudes and α the direction of P. Assume the sense of Q as per the next to the last paragraph of § 46.

Write the B equation, taking the center on the line of action of P, and establish the magnitude and sense of Q (§ 41).

One point only in the line of action of P being given, that point must, of course, be the center selected.

In writing B, memoranda may advantageously be inserted near each term, showing the value of each component of each force, as well as the moment of each component.

Write the A_x and A_y equations, inserting the value of Q just established, and get the horizontal and vertical components of P and establish P and α by routine like that for a similar purpose in Case 1.

Observe that the memoranda inserted near the terms of the B equation can be utilized in the A_x and A_y equations, just as the reverse was done in Case 1.

Observe also that Q once fully known, the problem is simply that of finding an equilibrant whose point of application is known in advance, and the work is thenceforth that of Case 1 for finding the magnitude and direction of the equilibrant.

Graphic Solution (Fig. 10 and Pl. III). Let the knowns be lettered *ab* . . . *cd*, and the unknowns accordingly *de*

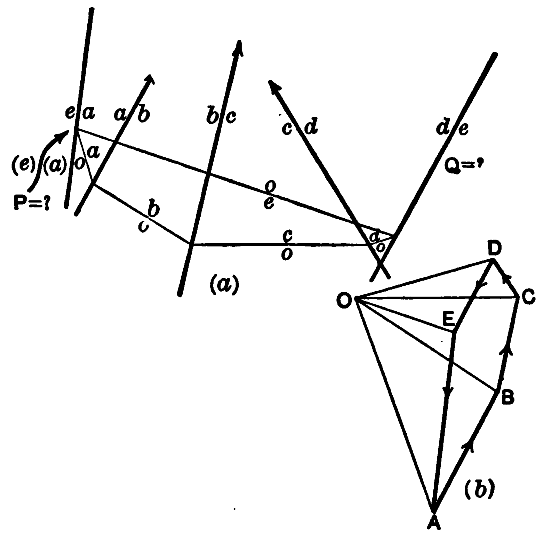

FIG. 10.

and *ea*, the former being unknown in magnitude (and sense) and the latter in magnitude and direction.

Construct the magnitude polygon from A to D, inclusive (Fig. 10*b*). Assume a convenient *oa*, **draw it through the given point of application** of *ea*, and construct the string polygon to *od*, inclusive (Fig. 10*a*). In order that *ao* may coincide with *oa* and close the string polygon, *oe*, the missing string, must intersect *ea* where *oa* does, i.e., at the only known

point of *ea*; furthermore *oe* must start from the intersection of *od* and *de*. Hence *oe* is located and the string polygon is closed.

E must then lie on a ray from O parallel to *oe*, and on a line from D parallel to *de*. E is thus located, and connecting E and A, the magnitude polygon is closed. The required magnitude (and sense), and magnitude and direction, are DE and EA respectively.

Observe that the key to this solution lies in locating *oa* in the only way possible, so that its point of intersection with *ea* may be known; i.e., by drawing it through the only point through which *ea* is known at the outset to pass.

50. Solution of Case 4. (Cf. Pls. IV and V.)—Required the magnitudes (and senses) of three forces in a set in equilibrium.

Algebraic Solution. Let P, Q, and R be the required magnitudes. Assume senses as per the next to the last paragraph of § 46.

The A_x, A_y and B equations can now be written, taking the center for B anywhere whatever, as in Pl. IV, and there result three simultaneous equations, whence the desired quantities can be evaluated. This method is needlessly laborious, and the simultaneous equations can always be avoided by the aid of the three moment equations of § 41, as follows.

Write three equations of moments, taking the centers at the intersections respectively of Q and R, R and P, and P and Q. These equations will yield the magnitudes (and senses) of P, Q, and R respectively.

Obviously the solution is indeterminate if P, Q, and R are concurrent or parallel.

In Pl. IV is worked out a general case, and the determination of the coordinates of the required center of moments is seen to involve considerable labor. Case 4, in its occurrence in practice, however, almost invariably appears with the

three unknowns so situated that the coordinates of the three centers are much more easily determined than in the problem of Pl. IV. The arms of the forces can usually be read directly from the drawing or can be calculated with little labor. In such cases the equations will best be written with the arms inserted directly without resort to components except for the more inconveniently lying forces.

A fairly typical example of Case 4 as it occurs in practice is fully worked out in Pl. V.

Sometimes two of the unknowns, say P and Q, are parallel. R can then be found by taking a center on the line of action of Q and off the line of P and R, bringing into the equation the previously calculated value of P.

Graphic Solution (Fig. 11 and cf. Pl. IV). Let the knowns be lettered *ab* . . . . *cd*, and the unknowns *de*, *ef*, and *fa*, the three last being unknown in magnitude (and sense).

Construct the magnitude polygon from A to D, inclusive (Fig. 11*b*). The two apices E and F remain to be located. That means that two strings *oe* and *of* are to be located, while the direction of neither is known, nor is the point in which they will intersect *ef* known. One of these directions might be assumed and the string polygon closed accordingly; the same result could be attained by assuming the point in which they intersect *ef*. The result of this process would in general be that the magnitude polygon would not close, and the labor would have been in vain.

Now, if either of these two strings could be made to vanish, i.e., to be of zero length, it could be looked upon as having any direction whatever, including a direction consistent with the closure of the magnitude polygon; then only one string remaining to be located, there would be no further difficulty. But any string, *om*, common to two forces, *lm* and *mn*, will vanish if *ol* and *on* intersect at the point of intersection of *lm* and *mn*.

Accordingly if *oa* be drawn (Fig. 11*a*) at the outset through the intersection of *ef* and *fa*, on proceeding with the string

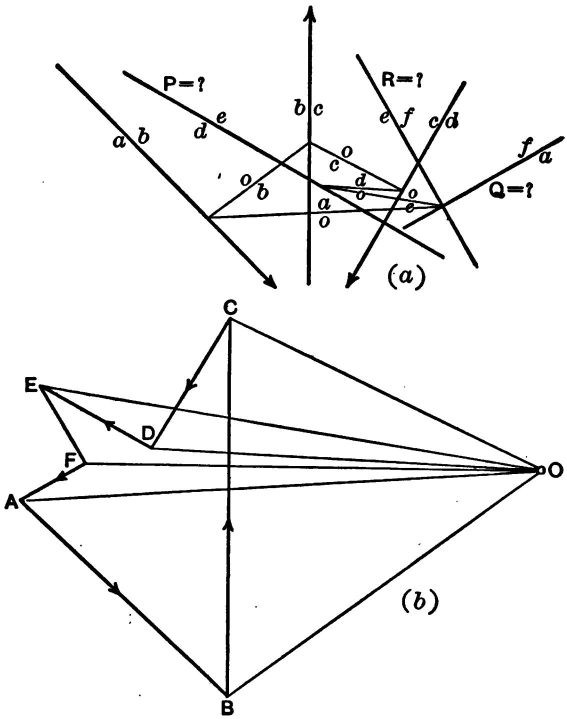

FIG. 11.

polygon *oe* will fall into place, connecting the point where *od* cuts *de* with the intersection of *of* (whatever its direction) and *ef*. The string polygon thus being closed, *E* is located at the intersection of lines from *D* and *O* parallel respectively to *de* and *oe*. *F* can now be located at the intersection of lines from *E* and *A* parallel respectively to *ef* and *af*. The magnitude polygon is now closed, the direction of the ray *OF* is, of course, perfectly consistent with the string polygon being closed, and the required magnitudes and senses are *DE*, *EF*, and *FA*.

Observe that the key to this solution lies in drawing the first string through the point of intersection of two properly selected consecutively lettered unknown forces, and that the problem could also have been solved by starting *od* at the intersection of *de* and *ef*, causing *oe* to vanish, *F* to be located by aid of the string *of*, and *E* by parallels to *de* and *ef* from *D* and *F* respectively.

Observe also that the graphic, unlike the algebraic, solution, is not essentially more laborious in an instance like that of Pl. IV than in Pl. V.

50a. Remarks on Cases 3 and 4. Case 3 may be looked upon as a combination of Cases 4 and 1, for any two convenient components of the force unknown in magnitude and direction might be substituted for this force. Then they and the other unknown could be handled by Case 4, and the two components combined into the single required force by Case 1. In fact it not uncommonly happens that the two components are as acceptable a result in practice as their resultant and some labor is saved by substituting them.

In a similar way Case 4 may be looked upon as a combination of Cases 3 and 2*a*. The intersection of the two unknowns may be regarded as the given point of application of the resultant of the two unknowns, which can then be found by Case 3, and resolved into its components, the required forces, by Case 2*a*. This view is of little value in practice, unless perhaps in throwing additional light on the graphic method for solving Case 4 given in the preceding section.

Exercise 8. Sketch carefully free hand the graphic solution of each of the four cases, assuming the necessary data for each.

Exercise 9. Same as Exercise 8, but done to scale as usual.

Exercise 10. A derrick mast, supported by the usual socket and guys, is 40 ft. high and the boom is 60 ft. long. The boom is held at an inclination to the vertical of 30 degrees by a stay running from its upper end to the top of the mast. If a weight of 5000 lbs. be suspended from

the end of the boom, what would be the forces transmitted through the boom and stay?

Solve graphically and algebraically, in the latter case using the A_x and A_y equations, and also by two moment equations of § 41.

Exercise 11. A uniform beam rests upon supports at the ends at the same level. The length of the beam is 20 ft. It carries vertical loads of 10, 20, 30, 40 cwt. at 5, 8, 13, and 16 ft. respectively from the left ends. Its own weight is 5 cwt. Determine the reactions R_1 and R_2 at the ends. Solve by both methods.

Suggestions. Draw the data diagram to the scale 1 in. = 5 ft. One data diagram serves as usual for both methods of computation. A single heavy line will suffice for representing the beam.

Letter forces in the space diagram in continuous circuit around the bar.

For the algebraic solution apply § 41, determining R_1 and R_2 each by an independent equation in the form given in Pl. II*b*.

R_1 and R_2 once determined, *check by seeing if A_x and A_y are satisfied.*

Exercise 12. A body in the shape of an isosceles triangle is supported by a smooth hinge at one of the equal angles and by a smooth horizontal plane (at the same level as the hinge) at the other. Assume three unequal parallel forces normal to the slope on the side next the hinge to be given, and determine fully the pressures on the hinge and plane. Solve by both methods.

Exercise 13. P_1, P_2, P_3, P_4, and P_5 (Fig. 12) are known forces acting on the body shown, and Q, R, and S are known only in line of action. If

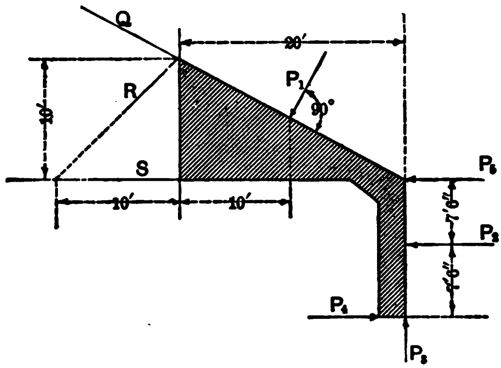

FIG. 12.

the magnitudes of the given forces are 4000, 3000, 2000, 1200, and 1500 lbs. respectively, determine the magnitudes (and senses) of Q, R, and S by both methods.

Suggestions. Construct the data diagram carefully. Scale 1 in. = 5 ft.

See Pl. IV for solution by the usual A_x, A_y, and B. Observe what labor this solution involves.

Use the method of § 41 and solve, scaling arms from the drawing only for checking. In writing the equation it will be convenient to write some of the moments in the form $P \times a$, and some in the form $P \cos \alpha\, y - P \sin \alpha\, x$. (Cf. Plate V.)

Exercise 14. A square plate (Fig. 13) weighing 900 lbs. is held in a vertical plane, with its edges inclined 30° to the horizontal and vertical,

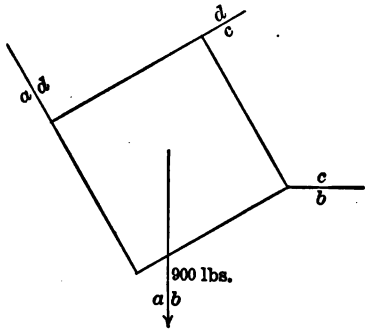

FIG. 13.

by a horizontal force and forces along two adjacent sides as shown. Determine these three forces graphically only.

Suggestions. Though this is clearly Case 4, note and use the short cut possible in this special case where there are two pairs of forces concerned whose resultants are necessarily equal, opposite, and coincident. Putting in this common resultant—which will naturally be lettered *ac*—there are two groups of concurrent forces, *ab*, *bc*, and *ac*, and *ac*, *cd*, and *da*. In the first *ab* is known, whence *bc* and *ca* follow by Case 2*a*; *ac* can then be resolved similarly into *cd* and *da*.

Note that what makes this exercise a special case is that there is only one known force. The method just pointed out may be regarded as the standard method of resolving a force graphically into three components.

The algebraic method if called for would have employed the three moments equations as usual in Case 4.

What could be specified as to the position of the pole in the general method of Case 4 which would lead to a solution identical with this short cut? How are all four sides of the string polygon then accounted for?

Exercise 15. Resolve (both graphically and algebraically) a given force of any convenient assumed magnitude into two parallel components whose points of application are

(*a*) On opposite sides of the given force.

(*b*) On the same side of that force.

Suggestions. Avoid a small scale for the space diagram. As is usual in this Case, the equations of moments are to be preferred to the A_x, A_y, and B equations for the algebraic work.

NOTE.—For further general problems in statics, the reader is referred to such works as Loney's Statics, Bowser's Analytical Mechanics, Minchin's Statics, Walton's Problems, etc. The reader seeking practice among the problems there given must be prepared in many of those problems for a large amount of geometrical analysis and trigonometric reduction extraneous to the purely statical solution.

CHAPTER VII.

ADDITIONAL GENERAL TOPICS AND PROCESSES.

51. Graphic Representation of the Moment of a Force.—THEOREM. If, through any point, a line be drawn parallel to a given force, P, the distance, y, intercepted from this line by the two strings belonging to the force is a length such that when multiplied by the force H, measured by the perpendicular dropped from O upon the given force in the magnitude diagram, the result will be numerically equivalent to the moment of P about the given point.

Proof.—Let m (Fig. 14) be any point whose perpendic-

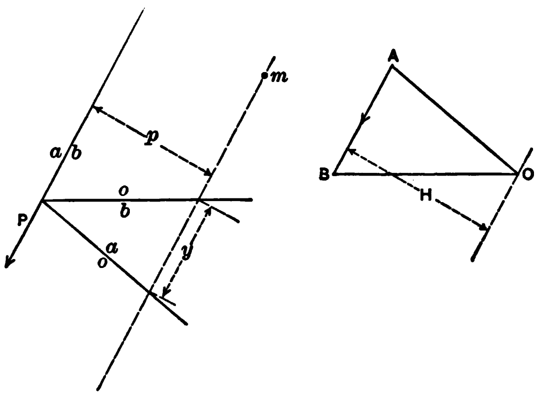

FIG. 14.

ular distance from the line of action of P is p. If P be lettered ab, and the pole be taken and the string polygon be constructed as shown, y is a length such that

$$Hy = Pp.$$

For, noting the similarity, by construction, of the triangles OAB and the one bounded by oa, ob, and the line through m, it appears that

$$AB : H = y : p,$$

or as AB represents the magnitude of P,

$$Hy = Pp. \qquad \text{Q. E. D.}$$

The intercept y may accordingly be said to **represent** the moment of P about m, it being understood that, to get the numerical value of this moment, y measured in the proper scale of lengths must be multiplied by H measured in the proper scale of force magnitudes.

So far H and y are simply a force and a distance whose product is the same as that of P and p. There are of course an indefinite number of forces and distances whose products will have this value, and the substitution if desired might be made in simpler ways even than by this theorem, if that were all that is desired. With a series of parallel forces, however, treated as usual with the magnitude and string polygons, H will be constant for them all, and aided by this fact the theorem leads to a convenient graphical method for the treatment of moments which will be developed in the next section.

Observe that the strings oa and ob are simply any two components of P, and that H is merely their common component normal to P.

52. String Polygon for Parallel Forces a Diagram of Moments.—As a corollary to the theorem of the preceding section, it may be stated that with parallel forces any two sides of the string polygon, extended if necessary, will cut from a line parallel to the forces an intercept which will represent the sum of the moments (about any point in that line) of all the forces at the apices of the polygon included between the two sides.

For example, suppose, in Fig. 15, the string polygon be drawn for any set of parallel forces (magnitude polygon not shown) which for the sake of generality are not taken in equilibrium and which are lettered preferably, though not necessarily, in the order of their occurrence. Lengths are intercepted on the line *MN*, parallel to the forces, which represent the

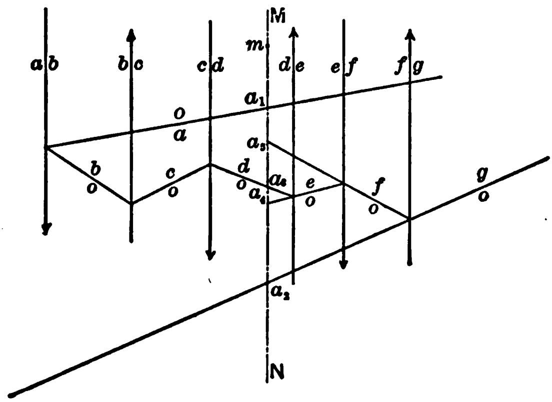

FIG. 15.

moments of groups of these forces about any point *m* on *MN* as follows.

The intercept a_1a_2 bounded by *oa* and *og* represents the moment of *ag* by direct application of § 51; that is, it represents the moment of the resultant of the whole set, and hence the sum of the moments of all the forces of the set.

Similarly, a_1a_3 represents the moment of *ad*; that is, the sum of the moments of *ab*, *bc*, and *cd*. Likewise a_2a_3 represents the moment of *dg*; that is, of *de*, *ef*, and *fg* combined.

Moreover, the intercepts are the actual summations of other intercepts each representing the moment of one of the individual given forces. Take, for instance, the last case, that of *de*, *ef*, and *fg*. Extending the strings *oe* and *of* till they cut *MN*, the resulting intercepts a_3a_4, a_4a_5, and a_5a_2 are seen by

§ 51 to represent the moments of *de*, *ef*, and *fg* respectively about *m*, and what is more, a_2a_3 is their algebraic sum $(-a_3a_4+a_5a_4-a_5a_2)$, as it should be.

The force *H*, it may be repeated, by which the lengths of these various intercepts must be multiplied to give the numerical values of the respective moments, is the force represented by the perpendicular (measured in the same scale as the rest of the forces) dropped from *O* upon the straight line *ABCD*, etc., upon which fall the parallel forces *ab*, *bc*, *cd*, etc., when they appear in the magnitude polygon. *H* being thus a constant force, the moments are proportional to the intercepts as above stated, and the string polygon for the parallel forces is a diagram of moments.

53. Remarks on the String Polygon as a Diagram of Moments.—Referring still to Fig. 15, it will be useful to note the following facts and deductions.

If the forces had been in equilibrium, *og* and *oa*, the strings including the whole set, would have coincided and there would have been no difference between a_1a_3 and a_2a_3, except in sign, and their sum would be found to be zero, as it should be.

If the forces had been reducible to a couple, *oa* and *og* would have been parallel and a_1a_2 constant, as it should be, for the constant moment of a couple.

Where *oa* and *og* intersect, a_1a_2 would vanish, as it should do, for that intersection is known to be on the line of action of the resultant of the set of forces.

If the forces in the given set are not parallel, the string polygon can still be made to yield the sum of the moments about a point of any consecutively lettered group of these forces. It is only necessary to draw a line through the point parallel to their resultant, scale the intercept from this line by the strings inclosing the group, and multiply it by the *H* for this particular resultant. The process is a straightforward application of

§ 51. Since in general no two resultants will have the same H, or be in the same direction, the string polygon for non-parallel forces is not a diagram of moments in the same useful sense as when the forces are parallel.

Hence the string polygon is not only an equivalent to an equation of moments in its scope and uses, but also an exact parallel to it in that it readily gives the value of any term of that equation and is simply a graphical means of finding and summing the values of those terms.

If any set of forces whose magnitude polygon closes is not reducible to a couple, the intercept from any line whatever in the plane by the two extreme strings must be zero. This could not be unless these two strings coincide, or, in other words, unless the string polygon close, a fact the converse of which could be taken as a demonstration of the preclusion of rotation by the closure of a string polygon alternative to that of § 32.

If the set of forces is reducible to a couple, the resultant looked upon as a single force is indeterminate in direction (cf. Exercise 7). Nevertheless its moment can still be found from the string polygon by assigning it any convenient direction, taking the intercept by the extreme strings (lettered, say, *oa* and *on*) from a parallel to it and multiplying this intercept by the H measured by the perpendicular from O dropped upon a parallel to the intercept drawn through the two coinciding points A and N in the magnitude polygon. The process by which the result was evaluated in Exercise 7 may be regarded as really an application of this method, in which a direction normal to the extreme strings was assigned to the resultant, making H identical with OA.

54. To Pass a String Polygon Through Three Given Points.—A process called passing a string polygon through three given points is sometimes useful in the study of the equilibrium of structures. It consists in locating a pole for a set

of two or more forces so that three specified strings of the resulting string polygon will each (prolonged if necessary) pass through one of three given points not in the same straight line.*

Problem.—Suppose, for example, that the forces in Fig. 16,

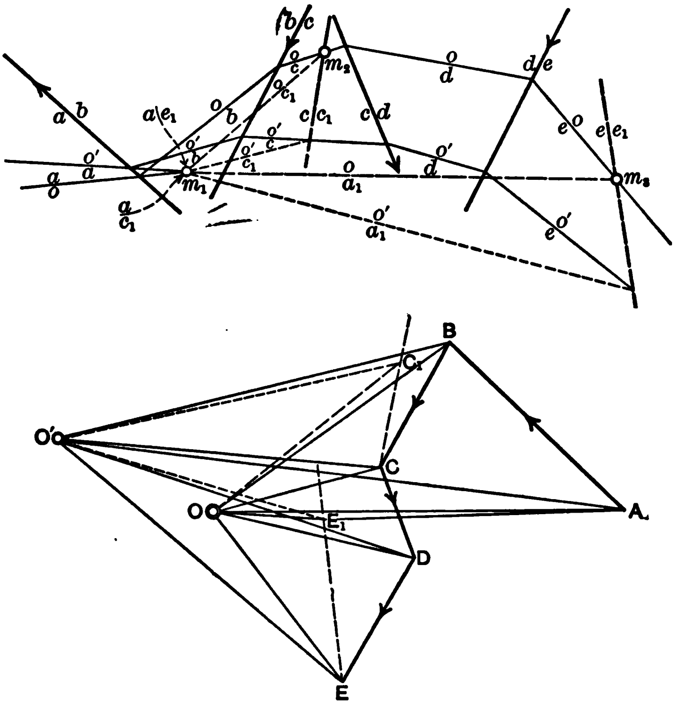

FIG. 16.

ab, *bc*, *cd*, and *de*, their magnitude polygon *ABCDE*, and the three points m_1, m_2, and m_3 be all given, and that it is required to select a pole O so that *oa*, *oc*, and *oe* will pass respectively through m_1, m_2, and m_3.

Solution.—Taking a pole O' at random, determine (Case 3) two forces CC_1 and AC_1, the former acting at m_2 in any

*If there is only one force in the given set, the string polygon can still be made to pass through three points not in a straight line, but two of the points will of course have to be on one of the strings.

convenient direction, and the latter at m_1 in a direction determined accordingly, such that they and the given forces *ab* and *bc*, intervening between m_1 and m_2, will constitute a set in equilibrium. Now, when *oc* passes through m_2 as required, oc_1 will have to pass through m_2 and m_1 in order to close the string polygon for this balanced set. It follows, moreover, for the same reason, that if oc_1 passes through m_1 and m_2, *oc* will have to pass through m_2, and working backwards, *oa* will have to intersect oc_1 again at m_1. Therefore an O taken anywhere on a line through C_1 parallel to m_1m_2 will be one for which, if *oa* be drawn through m_1, *oc* will pass through m_2.

Repeat this process, determining an AE_1 and an EE_1 at m_1 and m_3 respectively, which would balance *ab*, *bc*, *cd*, and *de*,— AE_1 and EE_1 corresponding exactly to the AC_1 and CC_1 respectively of the preceding step. Then reasoning as before, an O taken anywhere on a line through E_1 parallel to m_1m_3 will be one for which, if *oa* be drawn through m_1, *oe* will pass through m_3.

Therefore the intersection, O, of lines from C_1 and E_1 parallel respectively to m_1m_2 and m_1m_3 will be the required pole for which if *oa* be passed through m_1, *oc* will pass through m_2 and *oe* through m_3 as required. The string polygon *oa*, *ob*, *oc*, *od*, *oe* can now be constructed at leisure.

The foregoing process may be described in general terms as follows.

If for a set of given forces, *ab*, *bc*, *cd* . . . *yz*, it be required to pass any three specified strings, *od*, *ol*, and *os*, of their string polygon respectively through the given points m_1, m_2, and m_3 not in one straight line, draw a string polygon at random from a pole O' selected at random, beginning with the string *o'd*, making that pass through m_1, and by working both ways complete this random polygon at least as far as *o's*. Use this string polygon to determine (Case 3) ll_1 and dl_1 and ss_1 and

ds_1 (ll_1 and ss_1 being assumed to act at m_2 and m_3 respectively in any convenient direction, and the other two forces at m_1 and determined accordingly), which would balance $de \ldots kl$, and $de \ldots rs$ respectively. DL_1 and DS_1 do not need to be drawn. Parallels to m_1m_2 and m_1m_3 through L_1 and S_1 respectively will locate the required pole O at their intersection. To draw the required string polygon, od will be drawn first and made to pass through m_1. Then working both ways, the polygon is completed.

The problem obviously does not admit of solution when the three points are in a straight line.

It is advisable to take O' so that the random string polygon will lie as well as possible out of the way of the final required string polygon.

The random string polygon can be obviated if desired by an algebraic computation of the magnitude and direction of such a force as OD of the preceding general problem. D being given, O can then be plotted directly.

To determine OD we need to observe that its point of application m_1 is given, and that the resultant of it and $de \ldots kl$, and of it and $de \ldots rs$ must pass through m_2 and m_3 respectively.

It will be well to consider od replaced by its horizontal and vertical components meeting at m_1, whose magnitudes and senses * can be determined from two simultaneous equations of moments for centers respectively at m_2 and m_3, the first one for the series of forces $de \ldots kl$ and the unknowns, the latter for the series $de \ldots rs$ and the unknowns.

Combining the two components (by Case 2*a*), the required OD is found, and O can be plotted.

*It will be found advisable, in order to avoid error in such simultaneous equations, to assume senses of the components arbitrarily and show them in the sketch of data subject to subsequent correction on the solution of the equation.

It will be highly desirable in practice to check this location of O by determining in a similar manner the magnitude and direction of SO. The centers will be at m_2 and m_1 and included in the equations, besides the two unknown components will be the sets $sr \ldots ml$, and $sr \ldots ed$ respectively.

The object of the whole process is sometimes merely the determination of forces at two of the points, such as the OD and OS of the preceding work. The foregoing section constitutes therefore a complete description of the solution of such problems by both the algebraic and graphic methods. (Cf. § 86.)

Exercise 16. Assume any set of seven vertical forces uniform in sense and letter them in the order of their occurrence $ab, \ldots hi$. Assume three points m_1, m_2, and m_3 not in the same straight line and at the left of ab, between de and ef, and at the right of hi respectively. Pass a string polygon for the assumed forces through the three assumed points so that oa, oe, and oi will go through m_1, m_2, and m_3 respectively. Solve graphically only.

Remark. Assuming the set of forces as specified does not interfere with the generality of the method required and will save some labor.

55. An Alternative View of the Rays and the String Polygon.—Each two successive rays may be regarded as two components of the force with which they form a closed force triangle, and, moreover, by reversing them in sense, as components respectively of the forces next preceding and following that force in the magnitude polygon. Inserting these pairs of components in place of their respective resultants, there will result a set equivalent to the given set. If by moving the assumed point of intersection of the components along the lines of action of their resultants the pairs of equal and opposite components of the consecutively lettered forces can all be made coincident, the components must be a set in equilibrium, and hence the original set must have been as well.

The fact that the components were pairs of equals as well as opposites could only be if the magnitude polygon closed.

The pairs all coinciding form the closed string polygon. The lines of action of pairs of coinciding forces constitute the strings.

If the pairs of equal and opposite components all coincide and neutralize one another but one, that pair of components constitutes a couple to which the given set is reducible.

If the set is not reducible to a complete set of pairs of equal and opposite components (a case which can only arise when the magnitude polygon fails to close), there will be only one pair not of this character. The pairs of equals and opposites could be made to coincide throughout in the space diagram as usual, leaving the other two to locate their resultant passing through their intersection, a resultant likewise that of the given system.

From the preceding three paragraphs could be deduced anew the principle that the closure of the magnitude polygon and the string polygon insures equilibrium.

PART II.

APPLICATIONS.

CHAPTER VIII.

CENTERS OF GRAVITY.

56. Center of Gravity.—The **center of gravity** of a body is that point through which the resultant of the gravity forces acting upon all parts of the body will pass, whatever the attitude of the body, or what amounts to the same thing, whatever we may regard the slope of the gravity forces, so long as they remain parallel.* The center of gravity may be looked upon, then, as the invariable point of application of the force which represents the weight of the body.

Since the moment of the weight of the body, referred to any plane, is evidently equal to that of the weights of all the parts

* Strictly speaking, the center of gravity is a point in which the whole mass of a body might be concentrated without affecting gravity attractions existing between the body and all other bodies, whatever may be their relative distance and position.

Very few bodies have a true center of gravity. What is commonly called the center of gravity would more properly be called center of mass or center of area.

All centers of gravity are also centers of mass, but all centers of mass are not, strictly speaking, centers of gravity.

The term center of gravity is so thoroughly fixed by general usage in the sense above given, and the need for making the distinction is so rare, that there is no great need for urging the substitution of the more accurate terms. For further discussion of this distinction see Du Bois, Mechanics, vol. 2, chap. 4.

referred to the plane, we can write (conceiving gravity acting parallel to the plane):

$$\bar{x}\Sigma w = w_1x_1 + w_2x_2 + \ldots + w_nx_n,$$

$$\left.\begin{array}{lr} \text{that is} & \bar{x}\Sigma w = \Sigma(wx) \\ \text{likewise} & \bar{y}\Sigma w = \Sigma(wy) \\ \text{also} & \bar{z}\Sigma w = \Sigma(wz) \end{array}\right\}, \quad . \quad . \quad . \quad . \quad . \qquad (1')$$

where $\bar{x}$, $\bar{y}$, $\bar{z}$ and x, y, z are the coordinates of the centers of gravity of the whole and of parts of the body respectively, and w the weights of these parts.

It is often required to find $\bar{x}$, $\bar{y}$, $\bar{z}$, and to do it we need only to divide the body into parts whose x, y, z can readily be determined, and apply from (1′)

$$\bar{x} = \frac{\Sigma(wx)}{\Sigma w}; \quad \bar{y} = \frac{\Sigma(wy)}{\Sigma w}; \quad \bar{z} = \frac{\Sigma(wz)}{\Sigma w}. \quad . \quad . \qquad (1)$$

Equations (1) show that for all planes or axes of reference passing through this center of gravity, i.e. for $\bar{x} = 0$, $\bar{y} = 0$, and $\bar{z} = 0$,

$$\Sigma(wx) = 0; \quad \Sigma(wy) = 0; \quad \Sigma(wz) = 0; \quad . \quad . \quad . \qquad (2)$$

as might have been foreseen.

It is often convenient to apply the term center of gravity to bodies irrespective of their weights or to those having no weights, as geometrical shapes, and planes and lines, meaning thereby the point which would be the center of gravity of the body if the body were of uniform density or had weight proportional to volume, area, or length. Any factor introduced to express weights in terms of volumes, areas, and lengths would cancel out of equations (1) and (2) and leave simply volumes and areas and lengths in place of weights, to be treated exactly like weights. In (1) and (2), then, might be substituted v, a, or l for w. In the following work attention will be devoted mainly to the center of gravity of areas, and they like

lines will preferably be referred to two rectangular axes, and z will disappear.

In the foregoing the term "system of bodies" (if the bodies be understood to be fixed relatively to one another) might be substituted for "body."

Statically the location of a center of gravity is the location of the point of application of the resultant of a set of parallel forces, a simple special case under Case I requiring only the familiar equations (of which (1) above is, of course, a derivative) and polygons.

In the case of simple and symmetrical bodies the result of applying equations or polygons can, of course, be seen at once by mere inspection, just as can sometimes be done with other statical problems.

The center of gravity of a straight line is evidently its middle point; of a rectangle is the intersection of normals to the centers of its sides; of any parallelogram the intersection of its diagonals; of a circle or ellipse its geometrical center; of a triangle the intersection of its medial lines. This opens the way for proof by the general method that the center of gravity of a trapezoid is the intersection of the line connecting the middle points of its parallel sides with the diagonal connecting points found by extending each of the parallel sides in opposite directions by an amount equal to the length of the opposite side, or at the intersection of two such diagonals. (Fig. 17.)

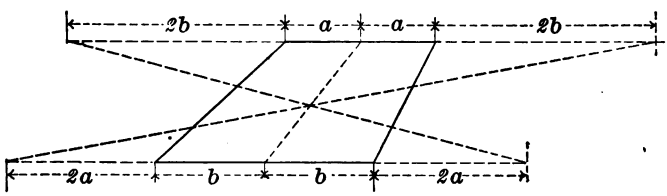

FIG. 17.

The center of gravity of a surface bounded by a curved line is, of course, treated by equation (1), but for strict solutions

in such cases subdivisions w are made infinitesimally small and the problem falls into the field of the integral calculus rather than arithmetic, without, however, introducing any new statical principle. The solution may be approximated with any desired degree of exactness by arithmetical methods by making the parts w smaller and smaller.

In the case of a segment of a parabola cut off by a chord normal to the axis, if b be the length of such chord and h its distance from the vertex, integration will show that its area is $\frac{2}{3}bh$, and that its center of gravity is on the axis distant $\frac{3}{5}h$ from the vertex (Fig. 18). The center of gravity of a circular

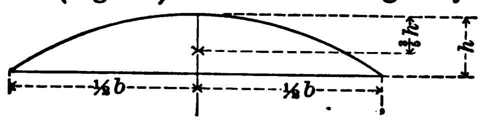

FIG. 18.

sector is shown by similar means to be distant from the center of the circle on the central radius by $\frac{2}{3}r\,\frac{\sin\,\theta}{\theta}$, where r is the radius of the circle and θ is *half* the central angle of the sector in circular measure.

Any plane surface which can be divided with sufficient exactness into triangles, rectangles, trapezoids, parallelograms, segments of parabolas, or sectors of circles can accordingly be treated by the general method. For a surface it is necessary in general to assume the gravity forces acting first in one direction, then in another, it may well be, at right angles to the first. The point common to the two resultants for the two different directions for the forces will be the center of gravity.

Numerical example. Required the location of the center of gravity of the irregular surface shown in Fig. 19.

Solution. Taking the origin at the lower left-hand corner, and dividing the figure into two rectangles and a semicircle, the areas and the coordinates of the centers of gravity of each of

these divisions are worked out and recorded in the figure for convenience in reference. The center of gravity of the semi-

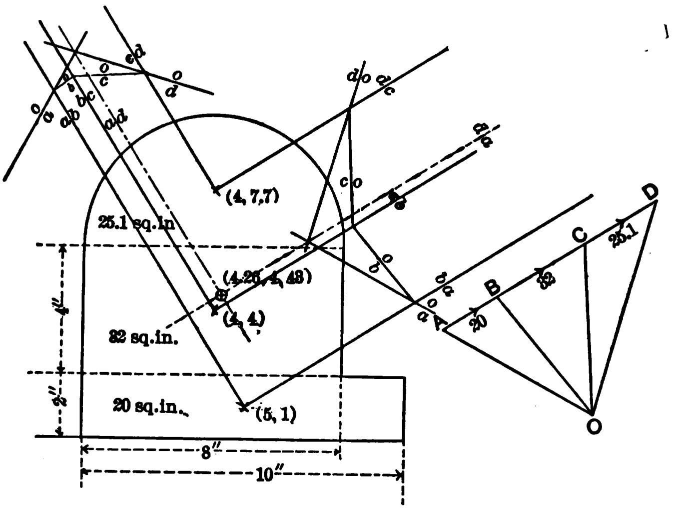

FIG. 19.

circle is, according to what has been said just above as to the center of gravity of a sector,

$$\frac{2 \times 4 \times \sin \frac{\pi}{2}}{3 \times \frac{\pi}{2}} = \frac{8}{4.71} = 1.70 \text{ in.}$$

above the center of the circle, making the ordinate of the center of gravity $2.0 + 4.0 + 1.70 = 7.70$ in. as shown.

Algebraic method. Calling $\bar{x}$ and $\bar{y}$ the coordinates of the center of gravity of the whole figure, the algebraic solution can conveniently be arranged as follows:

$$\begin{array}{ll}
25.1 \times 4.0 = 100.40 & 25.1 \times 7.7 = 193.27 \\
32.0 \times 4.0 = 128.00 & 32.0 \times 4.0 = 128.00 \\
20.0 \times 5.0 = 100.00 & 20.0 \times 1.0 = 20.00 \\
\hline
77.1 \times \bar{x} \;\; = 328.40 & 77.1 \times \bar{y} \;\; = 341.27
\end{array}$$

$$\bar{x} = \frac{328.40}{77.1} = 4.26 \text{ in.} \qquad \bar{y} = \frac{341.27}{77.1} = 4.43 \text{ in.}$$

Graphic method. Considering forces proportional to the areas of the divisions acting in two directions at right angles to each other from the centers of gravity of their respective divisions, and locating the resultant of each of these sets of parallel forces by the usual method, the required center of gravity is found at the intersection of the two resultants. This work is done in Fig. 19, with results agreeing with those found above. One magnitude polygon only is required. By transferring normals to its rays as well as parallels, both string polygons can be constructed.

The reader interested in a wider range of applications of the principles of this chapter is referred to works such as those mentioned at the close of Chapter VI, in addition to which might well be mentioned Rankine's Applied Mechanics, or Weisbach's Mechanics.

Exercise 17. Find the center of gravity of any irregular figure bounded by curves and straight lines. Solve both graphically and algebraically.

Suggestion. Select the figure from among the rolled-steel sections, such as the channels, unequal-legged angles, bulb angles, deck-beams, etc., given in the Carnegie, Cambria, Pencoyd, or other manufacturers' handbooks. Results may then be compared with those given in the handbook.

CHAPTER IX.

STRESS.

57. External and Internal Forces.—External forces upon a body are forces exerted upon that body by or through another body.

Internal forces in a body are those transmitted to one part of that body through another part of the same body.

Investigation of external forces determines whether a body as a whole is stable, without regard to whether all parts are strong enough to do what is required of them. Investigation of internal forces determines the strength required of the parts to insure the stability of the whole. A knowledge of both external and internal forces is therefore indispensable. Both internal and external forces are, of course, subject to the general laws governing forces.

The internal forces at any section of the body hold in equilibrium the external forces on either side of section.

58. Stress.—Stress is the tendency to distortion or rupture in a body due to the action of the external forces. It is a measure of the responsibility of the body or of a specified part of the body in service.

Stress may vary greatly in the various parts of the body, and it is necessary in any study of it to compute the stresses in existence at one or more plane sections of the body. In such cases the plane of the section will divide the body into two segments, and the external forces into two sets, one acting

directly on one segment, the other on the other. The two equal, opposite, and coincident resultants of these two sets of forces may be looked upon as the immediate cause of the stress and its measure as well. The magnitude of the stress depends upon the common magnitude of these two resultants, and its nature upon their other elements.

The determination of stresses, then, is simply the determination of elements of forces—a statical process pure and simple and one of the most important fields of application of statics. It will receive much attention accordingly.

59. Kinds of Stress.—The resultants mentioned in the preceding section may, of course, be either single forces or couples, and four special cases of stress are distinguished, each corresponding to a special direction for these single forces and to a special position for the planes of the couples.

With the single force resultants there is **Normal Stress** or **Tangential Stress** (more commonly called **Shear**) at the section according as the resultants are normal or parallel to the section. Their common magnitude measures each stress.

With the couples there is **Flexure** or **Torsion** at the section according as the couples are in a plane normal to the section or in planes parallel to it. The common magnitude of the couples measures the stress.

60. Combined Stresses.—As a matter of fact, however, the resultants are very frequently not in any one of these four simple relations to the section, and there results some combination of the four stresses accordingly. If the single force resultants are inclined to the section there exists both normal stress and tangential stress, each measured by the common magnitudes of the components normal and parallel to the section respectively. If such resultants fail to pass through the center of gravity of the section it will be convenient to observe that by § 20 each resultant is equivalent to a force acting at

that point* and a couple, and in such cases accordingly there would be combined normal stress, shear, and flexure. If the forces are non-coplanar, there may be still another couple in a plane parallel to the section, and torsion will be present in addition to the other stresses.

If there is normal stress or shear at the section as well as flexure, as is usually the case, the resultant couple which will measure the flexure is the resultant of the set made up of all the forces external to the body acting on the segment and also a force, internal to the body but external to the segment, equal, opposite, and parallel to these external forces conceived to be applied at the center of gravity of the section.†

In other words, flexure may be looked upon as measured by the couple remaining after providing the forces which will prevent motion of the segment normal or parallel to the section.

61. Further Particulars Relating to Stress.—Normal stress is divided into its more familiar subdivisions, **compression** and **tension**, according as the resultants act from their respective segments towards or away from the section.

Shear and Flexure are similarly subdivided according to the relative senses of the resultants, but their subdivisions have no established names, and are distinguished from each other by the adjectives positive and negative, arbitrarily applied. These distinctions are rarely needed except in the case of horizontal beams subject to vertical loads, in which the sections

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* Any other convenient point might be selected, but, as a principle of Resistance of Materials worth noting in passing, it may be said that there is no advantage in making the substitution of the force and the couple unless the center of gravity be the point of application of the force. It is known how to provide for a force so applied, but not for one elsewhere, except by this substitution.

† If this force be taken into account the value of the flexure would be independent of the center chosen, as the value of a couple is constant for all centers in the plane (§ 23). But taking the center of moments at the center of gravity the moment of this force vanishes and the force need not then be determined for the calculation of the flexure.

are taken normal to the axis of the beam, i.e. vertical. In such cases it is natural and customary to call shear and flexure positive when the left-hand segment is subject to an upward force or clockwise couple respectively, and vice versa.

Under normal stress a body simply tends to lengthen or shorten (according as the stress is compression or tension) along a line at right angles to the section. Compression is the stress typical of columns, posts, struts, and pedestals. Tension is typical of ropes and chains in service, or tie-rods. Anything through which a push is transmitted is in compression, and anything through which a pull is transmitted is in tension.

In Shear one segment of the body tends to slide by the other with no tendency to rotate about an axis normal to the section. Shear is the stress typical of rivets; in fact, their main purpose is usually to resist this kind of stress. The shearing resistance of the rivet in Fig. 20 prevents it from

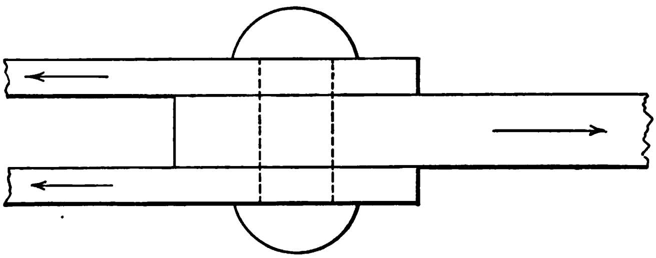


FIG. 20.

being separated into three parts by shearing on the planes of contact of the plates which it connects. Another illustration of shear is to be found (Fig. 21) in the plane indicated by the dotted line in the timber receiving thrust from the rafter.

Flexure is simply a tendency to bend. Familiar examples are a stick bent over the knee, or a loaded floor joist.

Torsion is a twisting tendency due to the two segments of

the body tending to rotate in opposite directions about an axis normal to the section. Torsion is the stress typical of shafting for the transmission of power. It is rarely permitted elsewhere if it can be avoided.

When it is said that Flexure or Torsion exists at a given section it means that the segment of the body on one side of

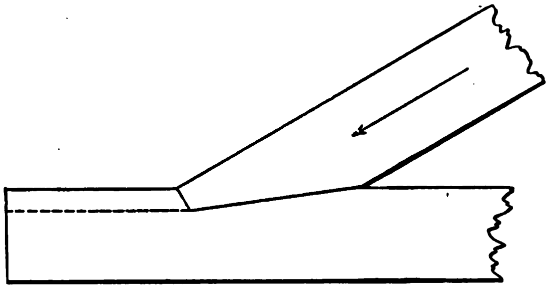


FIG. 21.

the section is likely to bend or twist respectively with respect to the other segment. These two stresses may be looked upon as complex cases of normal stress and shear respectively, i.e. cases in which the tendency to failure under the normal stress or shear varies in different parts of the section. It is familiar fact that the upper and lower surfaces of a bent beam and the surface portions of a shaft are subject to severer stress than the interior portions.

Familiar examples of combined stresses occur in beams subject to transverse loads in which shear and flexure appear together; in shafting in which shear, flexure, and torsion appear together, to which may be added compression if the shaft is vertical.

A stress is fully described as soon as its magnitude and nature are stated, nature being here understood to include algebraic sign as well as kind (§ 59).

CHAPTER X.

STRUCTURES.

62. Structures. Definitions.—Structures are simply artificial contrivances for supporting loads or resisting the active forces of nature. They must resist the stresses due to the action of these loads or forces on the one hand and the reaction of the earth's surface on the other. They may be very simple (as mere posts, pedestals, tension rods), acting under compression alone or under tension alone, and such may be designated by the term elemental structures. In general, however, the term structure will be understood to mean only the complex kind, such as are subject as a whole to bending and shear.

As regards their composition, structures may be divided into two broad types, (*a*) **framed,** and (*b*) **non-framed.**

A **framed structure**, or **frame**, is one composed of a series of straight bars fastened together by their ends only, so as a whole to make substantially one rigid body.

Ideally (that is, if the joints could be made frictionless hinges, if the weights of the bars could be made to act only at their ends, and if all other loads are applied only at the joints) the stress in each bar of a frame would be purely axial (i.e., pure tension or pure compression), making each member an elemental structure.

Such **ideal frames** are also called **true frames,** and are what is meant when frames are referred to without further description.

The nearest actual approximation to a true frame is a pin-

connected truss. Indeed pin-connections have been used very widely in truss construction largely because the assumed conditions of computation can thus be most nearly realized in practice (cf. § 63).

The resultant of the external forces at any section of a true frame is resisted by elemental members into which the section can easily be analyzed (for the center lines of the bars must be the lines of action of the forces internal to the structure) and each pin or joint is in equilibrium under the action of a set of concurrent forces.

A **non-framed structure** is one consisting of one continuous member or of a number of members so fastened together throughout their lengths as to make one solid piece.

A non-framed structure cannot be analyzed into separate elemental members. The flexure and shear at any section is resisted by the whole section under stress varying in intensity from point to point of the section according to more or less complex laws.

The relations between the external forces are entirely independent of whether the structure is framed or non-framed. The difference in the types is wholly one of internal make-up, and effects the method of dealing with internal forces only.

The periphery of a frame constitutes what are called the two **chords** of the frame, the portion of the periphery bounded by the end joints on the upper side of the frame being further designated as the top, or upper, chord and the one on the lower side as the bottom, or lower, chord.

63. Extent of Approximation to True Frames in Practice.—Outside of the fact that joints cannot possibly be made frictionless, and that the weights of the bars cannot possibly be made to act only at their ends, true frames are very uncommon.

Compression chords, that is, the large part of the periphery of every frame which is subject to compression, are usually

made stiff at the joints. Other members may be hinged to them, but the chords themselves are not broken at the joints, or are so rigidly spliced as to be practically continuous and hence stiff. Moreover, trusses are frequently made with no hinge joints whatever, but with tightly riveted connections.

The conception of the true or ideal frame as above defined is, however, fundamental to the design of all these structures, and is the hypothesis upon which the normal stresses in the members are determined. The difference between this ideal frame and actual ones can, by proper design and construction, be rendered negligible. When, however, the difference is permitted to be considerable, means are available for estimating and providing for the so-called secondary stresses thereby produced.

If the axes of the various bars intersect at common points, i.e., if the forces at a joint are really concurrent, secondary stresses are considered negligible in spite of the joint being far from frictionless. This is due to the materials being so nearly rigid that only very slight changes of shape in the structure occur and hence there occurs only a very slight tendency for the bars to turn about the joints. The forces themselves by meeting at the joint produce no tendency of this kind.

64. Loads Applied Elsewhere than at Joints.—It occasionally happens that loads must be brought to bear on frames at points where there is no joint, and where it is not practicable to make one by the addition of more bars to the frame. The result is that the bar to which the force is applied has to do double duty—that of any frame member subject to tension or compression and also that of a beam or girder subject to shear and flexure.

In its capacity as a beam the bar transmits its transverse loads to the frame in the form of reactions upon the joints by which the bar is incorporated into the frame. These reactions

are determined as in any case of a beam with given loads and supports. Reversing their senses, they are the forces acting on the joints of the frame. Making these determinations and substitutions the frame can be analyzed as usual. The shear and flexure for which (in addition to their tension or compression as part of the frame) the bars subject to transverse loads must be designed are determined as for any beam.

This form ot construction is uneconomical of material, and is usually to be avoided.

Strictly speaking, all bars of a material frame are in the condition just described owing to their own weight, but in frames of moderate size the resulting shear and flexure in individual bars is neglected even though the frame itself is analyzed for tensions and compressions due to its own weight considered concentrated at the joints.

For example of a frame illustrating this section see § 84 and Exercise 27.

65. Frames in General.—Frames may be (*a*) complete, (*b*) incomplete, or (*c*) redundant. A **complete frame** is one composed of just enough bars to insure its keeping its shape under all conditions of loading. If it has fewer bars than this

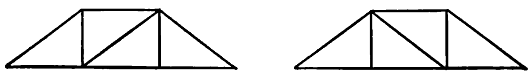


FIG. 22.

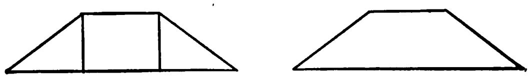


FIG. 23.

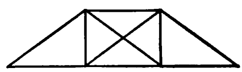


FIG. 24.

requires, but nevertheless is able to carry a load if properly distributed, it is **incomplete**, and if more, it is **redundant.** Examples of each are shown in Figs. 22, 23, and 24 respectively.

Since the triangle is the only geometrical figure in which a change of shape is precluded unless the lengths of its sides are changed, the triangle is necessarily the basis of arrangement of the bars in a frame.

A complete frame must accordingly be made up of the minimum number of bars consistent with its being composed wholly of triangles. Removing a bar common to two triangles would render it incomplete. Adding bars to a complete frame, as by adding the second diagonal in one or more quadrilaterals, renders it redundant, but if such bars are capable of resisting one kind of stress only, as in the case of counters, the redundancy may be only apparent. See § 83 and Exercise 26.

Complete frames are the type which is usually closely approximated, and which accordingly receives most attention.

Their analysis involves only a straightforward application of the familiar principles of statics.

Incomplete frames are stable only under symmetrical or other specially arranged loads. Under such loads they are analyzed with as much ease and certainty as Complete Frames, and require no further explanation.

Structures having outlines of incomplete frames may resist loads of any distribution by virtue of the flexural strength of members continuous through several joints, but such structures are not true frames and require special treatment, which takes note of the ability of some of their bars to resist bending.

Redundant frames * will resist loads of any distribution, and some forms are not uncommonly found advantageous in use. Their analysis involves statically indeterminate problems and

* A test for redundancy can be worked out as follows. Beginning with a triangle, each two bars added establishes a new joint. Then if $3 + x$ equals the number of joints, the number of bars for the complete structure will be $3 + 2x$, i.e., twice the number of joints minus 3. Therefore, for complete frames, if m equals the number of bars and n equals the number of joints, m equals $2n-3$. If m is less than $(2n-3)$, the frame is incomplete; if m is more than $(2n-3)$, the frame is redundant.

their stresses therefore cannot be worked out by purely statical methods, but with special data as to the material, or as to the perfectness of workmanship, or by the aid of some outright assumption as to probable action of the bars, a more or less satisfactory estimate of stresses can be made.

66. Loads.—Loads are (*a*) **permanent** or **dead**, those which are always present; or (*b*) **moving** or **live**, those which are only occasionally present.

The permanent load for roofs is the weight of the roof covering, purlins, trusses, etc.

The live load for roofs is wind pressure, snow, etc.

The permanent load on bridges is the weight of the floor, trusses, etc.

The live load on bridges is the weight of trains, carts, crowds of people, wind pressure, etc.

Loads which act simultaneously on a structure may or may not be considered all at once in the determination of stresses. If they are not considered all at once the total effect is obtained by simply taking the algebraic sum of all the stresses caused by the partial loads. It is usually desirable in practice to follow this course for the sake of avoiding very serious complications in the work.

The way in which stresses are provided for falls within the domain of Resistance of Materials.

67. Stresses in Structures.—As has been stated, stresses result directly from external forces. External forces are either loads or reactions. Loads are always known or assumed and the reactions determined accordingly by the methods already developed, commonly Case 2*b* or Case 3.*

* It may be noted that certain cases of statically indeterminate problems occasionally arise in connection with reactions. An example would be found in a beam resting on three or more supports, giving rise to three or more parallel reactions. The reactions can be worked out in such cases, by the aid of the laws of elasticity, involving methods outside the scope of this book. The need of resorting to such methods is usually avoided in the design of the structure.

As soon as the reactions are known, the external forces are all known and the stresses can be determined as a purely statical matter by methods to be inferred from the discussion of stresses in §§ 58–60. As there stated the stress at any section is really a question of the resultant of all the forces external to the body on one of the segments into which the section divides the body. This resultant is a measure of what may be looked upon as normal stress, shear, flexure, or torsion whether the body is framed or non-framed. Before either kind o structure can be designed, however, the statical analysis must go a step further still. In the case of non-framed structures this further analysis is statically indeterminate and requires the aid of the elementary principles of Resistance of Materials for its accomplishment. Stresses in a non-framed structure are accordingly considered determined as soon as each of the four kinds of stress is fully known at a requisite number of sections.

In framed structures this further analysis may be statically determinate. It consists in determining just how much tension or compression exists in each bar as a consequence of a given set of loads. The determination of stresses in a framed structure does not stop with the determination of normal stress, shears, flexures, and torsions in the body as a whole. In fact, in many cases it proceeds directly to the determinations of the tensions and compressions in the bars consequent upon the four fundamental stresses without stopping to find those stresses in their unanalyzed state at all.

The next two chapters will state the statical processes applied to each of the two kinds of structures after the external forces are all known.

CHAPTER XI.

STRESSES IN NON-FRAMED STRUCTURES.

68. Stresses in Non-framed Structures.—In a **non-framed structure**, stresses are found at any section by finding the resultant of all the forces on one side of the section. Its components normal and parallel to the section will measure the normal stress and shear respectively, and the senses of the components will decide whether the former is compression or tension and the latter positive or negative. Taking the intersection of the force with the section plane as its point of application, the moment of the normal component and of the parallel component about the center of gravity will measure the flexure and torsion respectively, and the signs of the moments will distinguish between positive and negative values of the stresses. This process is simply Cases 1 and 2*a* combined.

A process statically identical with the preceding is frequently more convenient in practice. In this process the components of the individual forces normal and parallel to the section are summed to get the normal stress and shear respectively, and the moments of the individual forces about the center of gravity of the section and about an axis through the center of gravity normal to the section are summed to get the flexure and torsion respectively.

In non-framed structures, stresses have to be worked out at only a relatively small number of critical sections. It is sufficient at other sections to be sure that the stresses do not

exceed certain amounts. Stresses existing at all sections can easily be shown, if required, as ordinates of properly constructed curves. (Cf. Exercise 18.)

Exercise 18. A horizontal bar is acted on by (20 240° 5, 0), (30 300° 8, 0), (40 90° 20,0), and by a force at (0, 0) whose line of action is inclined 30° to the horizon, and one at (12, 0) unknown both in magnitude and direction. Establish equilibrium and determine all the stresses at a section through (10, 0). Solve graphically only. Show clearly on the drawing how results were found and where they were scaled.

69. Shear Diagrams.—A diagram showing the value of the shear at all points of a vertically loaded beam can easily be constructed by selecting a horizontal base line and drawing parallels across the intervals between the adjacent forces at distances above or below the base line proportional to the magnitude of the sum of the forces on either side of the interval. Resultants of forces at the left of any segment indicate positive or negative shears and are set off upward or downward from the base line according as they are upward or downward.

Thus, in Fig. 25, the values of the shears in proceeding from left to right are in the four intervals respectively, $+51.77$, -48.23, -18.23, and $+31.77$. The shear diagram is consequently as shown in the lower shaded diagram.

The four numerical values just given are evidently the values of the resultants AB, AC, AD, and AE, and could have been projected across into their respective intervals from the magnitude diagram.

70. Flexure Diagrams.—Since the measure of flexure at any section is the sum of the moments about the section of all the forces on one side of the section, all that is necessary for obtaining a diagram showing the flexure at all sections of a beam under a set of parallel forces is to letter the forces in the order of their occurrence along the beam and draw their closed string polygon (§ 52). The intercept by this polygon from a line parallel to the forces is proportional to the flexure in the sec-

tion of the beam traversed by that line. The intercept needs only to be multiplied by the common H (§ 51)of the forces to give the numerical value of that flexure.

An example of a flexure diagram for a beam subject to a series of vertical forces is shown in the upper of the two shaded diagrams in Fig. 25. To find the flexure at any section all

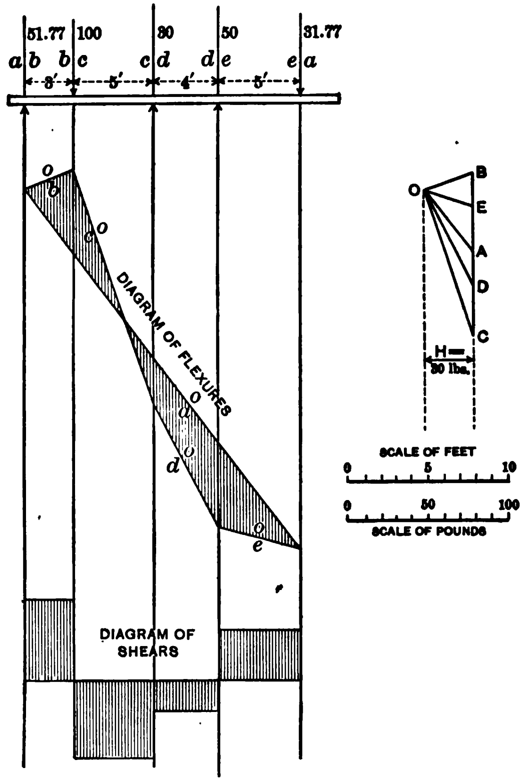


FIG. 25.

that is needed is to drop a vertical line from that section, measure the portion of this vertical within the diagram and multiply the resulting length by 30 lbs., which was selected as a convenient value for H in constructing the diagram.

Thus the intercept directly below the load *bc* scales 5.2 ft. The flexure in the section directly under *bc* is therefore $5.2 \times 30 = 156$ ft.-lbs. Algebraically the value is found to be more precisely $51.77 \times 3 = 155.31$ ft. lbs. Values can be obtained in like manner for any other section. It should be observed that the percentage accuracy of the graphical result is no greater than the accuracy of the scaling of the intercept. With small intercepts therefore the percentage of inaccuracy in the graphical result may be considerable.

Intercepts above *ao* indicate positive flexure and those below *ao* negative flexure.

Observe that the diagram shows at a glance that the flexure in the beam increases steadily to a positive maximum under *bc*, then decreases to zero at a point a little nearer to *cd* than to *bc* and continues to increase negatively, though more slowly, after passing *cd* to a negative maximum under *de*, when it negatively decreases rapidly again to zero under *ea*. The negative maximum is found to be numerically slightly larger than the positive maximum, and the dangerous section of the beam, so far as flexure is concerned, is under *de*.

Observe that the sections where the flexure reaches its greatest values, whether positive or negative, are those in which the shear passes through zero—a phenomenon of inevitable occurrence, as will be shown in the next section.

Exercise 19. A horizontal beam of 30 ft. span, supported at each end, carries loads of 300, 600, 1800, and 1200 lbs. at points 6, 10, 18, and 25 ft. respectively from the left end. Neglecting the weight of the beam itself, determine by both methods the numerical value of the flexure at sections 8, 12, 18, and 30 ft. from the left end. Record results side by side for comparison.

Suggestions. Take the scale of lengths as great as 1 in. = 4 ft. Take some convenient round number for the magnitude of *H*.

71. Connection Between Shear and Change in Flexure.— It will be useful to see if there is a simple relation between the

flexure at the end of an interval and the shear and flexure at the beginning of the interval.

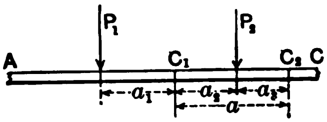


FIG. 26.

Accordingly, let AC, Fig. 26, be any segment of a beam, C_1 and C_2, and M_1 and M_2 the ends of the interval and values of the flexures existing there respectively, P_1 the resultant of all the forces at the left of C_1, P_2 of those in the interval C_1C_2. Dimensions as shown. Then

$$M_2 = P_1(a_1 + a_2 + a_3) + P_2a_3$$

and $$M_1 = P_1a_1.$$

Subtracting the latter from the former,

$$M_2 - M_1 = P_1(a_2 + a_3) + P_2a_3.$$

But P_1 is the measure of the shear at the beginning of the interval and $(a_2 + a_3)$ is the length of the interval. Calling the former V, the latter a, giving P_2a_3 the more convenient characterization $+ M_3$, the last equation takes the form

$$M_2 = M_1 + Va + M_3.$$

That is, **the flexure at the end of an interval is measured by the algebraic sum of the flexure at the beginning of the interval, the product of the shear at that section by the length of the interval, and the sum of the moments of the forces in the interval about the center of gravity of the end of the interval.***

Corollary. If a be of infinitesimal length, dx, M_3 will vanish, $M_2 - M_1$ may be written dM, and the equation takes the form

$$V = \frac{dM}{dx},$$

* The data of Fig. 26 are of the simple sort usual in the practical examples of this problem. The reader should satisfy himself that the same conclusion would have been reached whatever the directions of P_1 and P_2, or if one or both of them had been couples.

wnence it appears that the shear at any section is the x-derivative of the flexure at that section, and it follows that **where the shear passes through zero the flexure is at a maximum or minimum.**

Exercise 20. Shears and Flexures from fixed loads. Draw diagrams of shears and flexures for each of the seven following cases.*

(*a*) Cantilever with W concentrated at the outer end; (*b*) cantilever with W uniformly distributed; (*c*) simple beam with W concentrated at the middle; (*d*) simple beam with W uniformly distributed; (*e*) simple beam under several concentrated loads; (*f*) beam overhanging one support, concentrated load at the end of the overhang, two others between the supports; (*g*) beam with equal loads at each end, supports at equal distances from the ends.

Compute algebraically and compare the maximum shears and flexures in cases (*a*), (*b*), (*c*), (*d*).

Suggestions. Note that in (*a*), (*c*), and (*e*) the diagram of shear can be transferred directly from the magnitude diagram,—strictly also in (*b*) and (*d*).

In dealing with (*b*) and (*d*) construct approximate diagrams by dividing the load into short portions, and treat as a set of equivalent loads concentrated at the centers of gravity of these portions. Points in the true diagrams will lie at the intersections of the extended vertical boundaries of the portions with the approximate diagrams thus found and in the case of shear will be the straight line through these points; in the case of flexure will be the parabola inscribed in the approximate string polygon and tangent to it at these points.

* Here the term cantilever is used to designate a beam supported at one end only (by being built rigidly into a wall, for example) or in any way overhanging a support; a simple beam is understood to mean one resting upon supports at each end without constraint.

CHAPTER XII.

STRESSES IN FRAMED STRUCTURES.

72. Stresses in Framed Structures.—In a framed structure the resultant of all the loads on one side of a section can pass the section, so as to hold the other segment in equilibrium, only in the form of simple pushes and pulls which must act along the center lines of the bars cut. These bars should be imagined to be replaced by forces acting along their center lines, which are a set of forces into which the resultant can be resolved—a problem which is determinate if the number of forces does not exceed two or three. This resolution accomplished (Case 2 or Case 4), the compression or tension in each bar is known and the requisite stresses determined.

Here as with non-framed structures it is usually unnecessary actually to evaluate the resultant of the external forces on one side of the section. The individual external forces are treated as the given forces in Case 2 or Case 4, and the unknowns are the two or three components of the resultant which measure the required stresses.

This method could be applied repeatedly until every bar in the structure had been cut and the stress in it determined, but when the stresses in a large number or in all of the bars in the frame are to be found, a less laborious method suggests itself, when it is observed that each and every joint is in equilibrium under a set of concurrent forces. Selecting a joint if possible where only two bars concur with one or more external forces, the stress in these two bars can be found by Case *2a*.

The joint at the other end of one of these bars will then commonly be found to be subject to forces, only two of which are now unknown—one of those previously unknown being the one just established, for the force which a bar exerts upon a joint at one of its ends must be the equal and opposite of that exerted upon the joint at the other end. Proceeding thus, applying Case *2a* to joint after joint till all have been treated, the stresses in all the bars are known.

If there are no joints where so few as two bars concur, the problem is of course statically indeterminate unless the frame is so composed as to permit the determination of one of three concurring bars by using method of sections, § 73. (Cf. Exercise 25.)

Sometimes a similar step has to be taken to deal with joints in the interior frames of where three forces concur, all incapable of determination by the ordinary method of Case *2a*. (Cf. Exercise 24.)

73. **Method of Sections.**—Method of Sections * is the name applied to the method of determining stresses in a framework by dividing the structure into two segments by means of a section plane, treating the bars cut as mere lines of action of forces external to the segment, and finding magnitudes accordingly. The problem presented is Case 4, when three bars are cut by the section, or Case 2, in the rarer case, when only two are cut. The calculations can be made for whichever segment the work will be easier, or, if a check be important, for both segments.

* This method solved algebraically is sometimes more explicitly called Ritter's Method of Sections, after Professor August Ritter, of Aix-la-Chapelle, who used it freely in his *Dach- u. Brücken-Constructionen.* Solved graphically the process is also called Culmann's Method of Sections, after Professor Culmann, of Zurich.

It must be observed that the methods used in this book for finding stresses are really all methods of sections, and the limitations imposed in this section must be seen to be arbitrary or conventional, with a view to the establishment of a convenient technical term.

Wherever we can divide a structure without cutting more than three bars, if these three bars do not meet in a point, the stresses in the bars can be determined.

If stress in a single bar only is desired, one simply selects a convenient section plane which will cut this bar and only two other bars.

The algebraic method of solving Case 4 will usually be found more convenient than the graphic for these problems.

Lever-arms, however, are sometimes so troublesome to calculate that scaling them from a carefully made large scale drawing is the best way to get them. Writing moments in form $P\,(y \cos \alpha - x \sin \alpha)$ is always an available resource.

The reasoning underlying this method is still further explained by Fig. 27. Fig. 27*a* represents any frame in equilib-

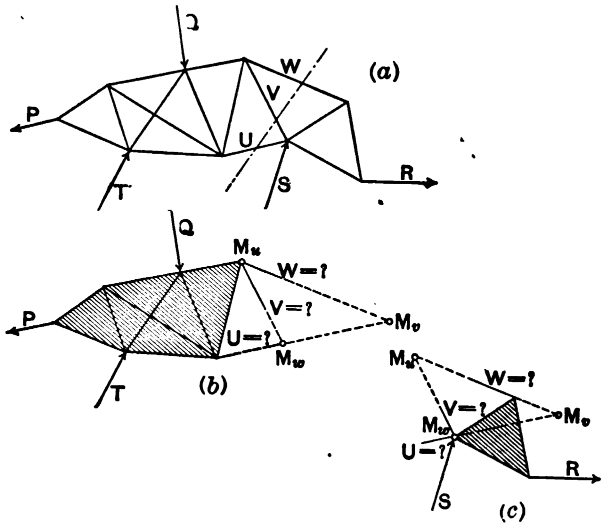


FIG. 27.

rium under the external forces P, Q, R, S, and T. It is required to find the stress in V. Intersecting the frame by a section cutting V and two other bars, U and W, we have as a

result what can be regarded as two rigid bodies shown shaded in Figs. *27b* and *27c* in equilibrium under the forces (external to them) T, P, Q, U, V, W and R, S, U, V, W, in either of which V can be established by an equation of moments with M_v, the intersection of U and W, as a center. If stresses in U and W be also required, they can be found by taking centers at M_u and M_w respectively.

Observe that this solution is entirely independent of the number or inclination of bars in the frame other than those cut, provided P, Q, R, S, and T remain unchanged in all their elements.

Plate V may be looked upon as giving a complete graphic as well as algebraic solution by this method, if the shaded body be considered the segment of a frame, and the three forces, P, Q, and R, the forces acting through the cut bars.

74. Method for Determining All the Stresses in a Frame Under a Given Load.—To determine all the stresses in a non-redundant frame, since it is made up of a series of sets of concurrent forces, we need only to

(1) See that the external forces are in equilibrium, and

(2) Work out case *2a* for each joint.

In the former of these two processes the algebraic method is generally to be preferred. It usually is less laborious, and the superior precision of its results is welcome, as the accuracy of all the succeeding work depends upon them.

In the latter process, however, the graphic method is greatly to be preferred, as saving much troublesome labor with a minimum risk of serious error with a degree of accuracy amply sufficient for the needs of engineering design. This work consists simply of beginning with a joint where two bars concur with one or more external forces, and working out a closed magnitude polygon for the set of concurrent forces so composed. Two of the sides of this polygon will determine

the stresses in the two bars—not only in magnitude, but also in nature, the former being shown by the lengths of the sides, the latter by the arrows upon them, for these arrows must follow one another around the polygon, and when transferred to the bars to which they belong they indicate compression or tension according as they point toward or away from the joint. Proceeding to the next joint, which, with the force acting through the first two bars known, will in turn have only two unknowns concurring upon it, a second polygon is constructed and so on through the whole series of joints. Thus would result a series of magnitude polygons, one for each joint.

75. Example.—An example is worked out in Fig. 28. The frame there shown, Fig. 28*a*, is known to be in equilibrium under the five external forces, 20, 200, 30, 140, and 110 lbs., through the first three having been given at the outset and the last two having been determined (Case 2*b*) from the dimensions of the frame and the positions of the supports.

It is most convenient to letter the external forces in the order of their occurrence around the outside of the figure and have the letter common to two adjacent forces apply also to the bar or bars in the periphery of the frame between their points of application. Letters added inside each triangle of the frame complete the lettering of the forces.

The work can begin at either of the two end joints. Taking the left end joint, we lay off EAB as the beginning of its magnitude polygon; then a line from B parallel to bf and one from E parallel to EF would locate F as in Fig. 28*b*. The polygon is in the order of the arrows, $EABFE$, and the stresses in bf and ef are consequently 127 lbs. compression and 111 lbs. tension respectively. Since ef and bf are now known, the polygon for the lower of the two joints next on the right can be completed by drawing from F and E parallels respectively

to *fg* and *eg*, locating *G*, and closing the polygon in which the arrows are in the order *EFGE*, showing the stresses in *fg* and *ge* to be 173 lbs. compression and 130 lbs. tension respectively.

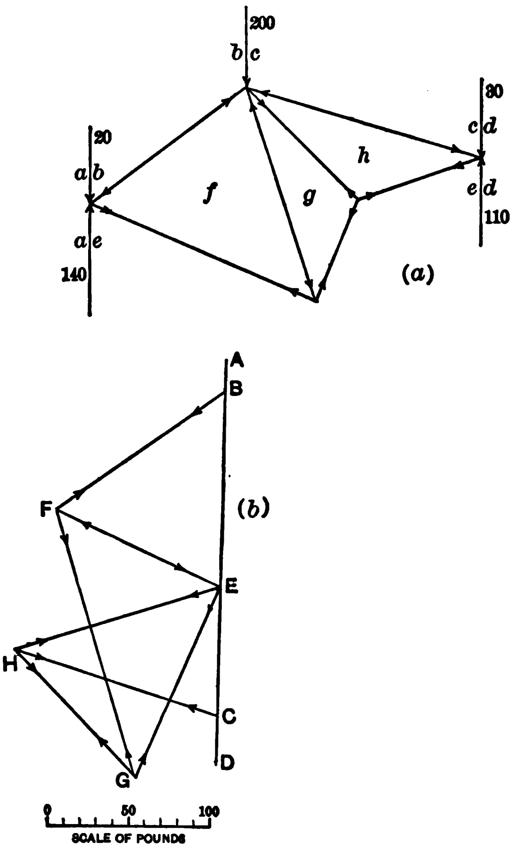


FIG. 28.

Only two forces are now unknown at the joint where *bc* is applied, and this polygon can be closed. The bar *fg* acted downward upon the preceding joint; it must therefore act upward at this joint. *GF* and *FB* of the polygon are already

drawn, adding the known BC, drawing CH and GH parallel respectively to ch and gh, H is located and the polygon is closed and consists, following the arrows, of $BCHGFB$, showing stresses in ch and gh to be 132 lbs. compression and 110 lbs. tension respectively.

For the joint at the lower end of gh only eh is unknown, and to close this polygon we have only to connect E and H. The arrows are in the order $EGHE$, and eh is 133 lbs. tension.

For the joint at the extreme right two of the forces, ch and eh, have been determined and cd and de were known at the outset. It is important as a check to see if they are in equilibrium. Laying off CD downward and DE upward, equal to 30 and 110 lbs. respectively, E is found to fall where it was before located, and the work checks with $CDEHC$, read with the arrows as the last magnitude polygon.

Examining Fig. 28*b*, it appears that it consists not only of a closed magnitude polygon for each joint, but also a complete magnitude polygon, $ABCDEA$, for the external forces. In fact, the ordinary method of determining all the stresses in a given frame under given fixed loads is, after equilibrium is established, to see that the external forces are lettered * in the order of their occurrence in passing around the outside of the frame, and complete their magnitude polygon before constructing any of the minor ones for the internal forces at the joints. Then adding two new lines to the diagram will establish the first joint, two lines more the next, and so on, each two lines completing a new magnitude polygon until the last joint is reached, where one bar only remains to be determined, and letters are already in the diagram which only need to be con-

* It should be pointed out that a letter which is common to two adjacent forces may be written once midway between them just as well as closely adjoining each, and is so written in practice.

nected to show this last stress. If the line so drawn is parallel to its corresponding bar the work checks, and the results can be scaled off as required. If the work does not so check, it shows that there is an error somewhere which must be found, and corrected before the result for any bar can be looked upon as trustworthy.

76. Stress Diagrams.—A diagram such as just described, including

(1). **A closed magnitude polygon for the external forces,** and

(2). **A closed magnitude polygon for each joint of the frame,** is called a **stress diagram**—sometimes also a Maxwell, or Cremona diagram.

In constructing stress diagrams as large a scale as is convenient should be used throughout, especially in the diagram of the frame, where otherwise short lines may give rise to inaccuracies when long lines have to be drawn parallel to them.

The equivalent of very large scale for frame diagram, without some of the worst disadvantages of such large scale, can be produced by drawing in a group long parallels to the bars through calculated points. These can then be used with a frame diagram of moderate scale or even a mere sketch for a guide.

If a number of stress diagrams are to be drawn from one frame diagram, natures of stresses from different loadings may well be recorded on small freehand diagrams accompanying each stress diagram.

To avoid confusion, the lines of action of the external forces should be shown entirely outside the periphery frame as in Fig. 28*a*.

Each line of the stress diagram, except those representing the external forces, represents the two equal and opposite forces which are in action at each end of the bar bearing the same letters as the line.

An error outside of common blunders in the use of draughting instruments, which is frequently the cause of failure of stress diagrams to close, is an incorrect determination of the reactions which gives rise to a set of external forces amounting to a couple instead of being in equilibrium, and such an error is not exposed by the magnitude polygon for these forces. The reactions should therefore be figured out independently, i.e., the correctness of one should not be allowed to depend upon the correctness of the other.

77. General Instructions Regarding Exercises Involving Stress Diagrams.—Determine reactions, whenever possible, by inspection, otherwise algebraically.

After completion of the stress diagram see that the work checks before proceeding further.

Show upon the diagram of truss, by means of algebraic signs, the nature of stress in each bar, using sign + for compression and — for tension.

As a guide in selecting a place for magnitude polygon of the external forces with a view to preventing work from running off the paper, it will be well to compute in advance one or two of the largest stresses, where the bars in which they will exist are easily discernible, as in Exercises 21–23.

78. Special Instructions Regarding Exercises 21–23.—State for comparison the graphic, algebraic, and semi-algebraic results for the bar specified, having used for the algebraic work the method of sections, and understanding by semi-algebraic work a similar algebraic solution in which lever arms are scaled from the drawing instead of being computed.

Note that checking semi-algebraically does not check the accuracy of the frame diagram.

Exercise 21. Truss shown in Fig. 29. Span 64 ft. Rise one fourth of the span. Eight equal panels; 1000 lbs. vertical load at each of the joints 0 and 8, and 2000 lbs. at each of the joints 1–7, inclusive. Scale

as large as 1 in. = 8 ft. Check the results for the bar 2–3 algebraically and semi-algebraically.

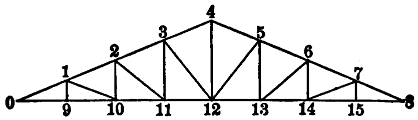


FIG. 29.

Exercise 22. Same truss and loads as in Exercise 16, but with additional loads of 4000 lbs. at joints 11 and 13, and 2000 lbs. at joints 9 and 15. Check the result for bar 10–11 algebraically and semi-algebraically.

Exercise 23. Same conditions as in Exercise 17, except that all loads at right of the center are doubled and 2000 lbs. added at joint 14.

Check the result for bar 3–12 algebraically and semi-algebraically.

CHAPTER XIII.

ADDITIONAL TOPICS AND EXAMPLES.

79. Complications in Connection with the Analysis of Frames.—The methods of the preceding chapter will be found easy of application to any simple frame under vertical loads. But many cases are unavoidable in practice where the conditions are not so simple, and a number of the most important and illustrative of them will be taken up in this chapter. The sources of difficulties are various. Those treated in the following pages fall into four classes, more than one of which may, of course, be found exemplified simultaneously in connection with one structure. These classes of the sources of difficulty are

1. Indeterminateness, apparent or real, as to the reactions.
2. Indeterminateness in the analysis of a frame, apparent but not real,—due to special systems of grouping bars.
3. Apparent but not real redundancy in framing.
4. Structure being partly framed, partly non-framed.

The really indeterminate cases here taken up are such only as can be brought within the scope of statical methods by making assumptions of such reasonableness and of such general acceptability in practice that the fact that the cases are statically indeterminate is usually ignored.

Structures generally recognized as redundant do not fall within the scope of this book.

80. Reactions Due to Non-Vertical Forces.—Indeterminateness as to the reactions occurs in the case of structures subject to sidewise or any non-vertical forces, such as roof-trusses, towers, etc., exposed to the wind. These reactions are, in general, two forces of which only the points of application are known, hence involving four elements unknown (two slopes and two magnitudes), and the problem of finding them is indeterminate. Results satisfactory in practice can, however, be obtained in any one of these three ways:

(1) By supporting one end upon rollers, assuming them frictionless, thus making one of the slopes known.

Rollers are frequently present anyway in large roofs to provide for expansion and contraction, and such roofs at once come under this method.

(2) By assuming that each wall resists half the horizontal component of forces, thus indirectly assuming two slopes.

(3) By assuming that both reactions will be parallel to the resultant of all non-vertical loads.

In (1) we will call to our aid a mechanical contrivance and make an assumption, and in (2) and (3) we make assumptions only.

Each of these ways, experience has justified as of sufficient correctness.

This kind of indeterminateness arises whenever the resultant of the loads has a component parallel to the supporting surface or, if the supports are in different planes, to one or both of the supporting surfaces. For example, in the case of a door or gate hinged to a vertical jamb, the door may be in equilibrium however its weight be divided between the hinges. In such a case, of course, if analysis were necessary, the assumption would be that either hinge may have to furnish the whole vertical support, or else assume it all on one or the other of the

hinges, and by setting this hinge a little high, make the assumption a certainty.

81. The Fink Truss.—The form of roof-truss shown in Fig. 30, and known in this country as the Fink truss, has some

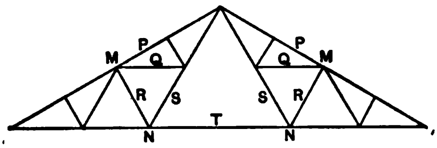


FIG. 30.

practical merits which cause it to be widely used. Though statically determinate it cannot be analyzed by the stress-diagram method without some special manipulation. On reaching either of the joints M or N, the three bars P, Q, and R, or R, S, and T, respectively are unknown, and the stress-diagram appears to be blocked. If any one of the five forces P, Q, R, S, or T can be evaluated the difficulty will be overcome. This suggests the method of sections, and it appears that a section can be taken through T either side of the ridge, which will cut only two other bars. T is determined accordingly.

In general in dealing with Fink trusses, the value of this stress might well be looked upon as something always to be computed in advance, as the reactions are, and to be laid off in place as soon as the magnitude diagram for the external forces is completed. Cf. Exercise 24.

Exercise 24. A Fink truss, which in this case will be made up of horizontal bars and bars inclined 30 degrees and 60 degrees to the horizon, as shown. Eight equal divisions in the upper chord. Load at each upper chord joint 2000 lbs. Construct the stress diagram.

82. Triangular Frame with Trussed Top Chord.—A frame somewhat similar to the Fink truss is that shown in

Fig. 31. It is a construction developed from the ordinary triangular frame by a simple system of trussing applied to the rafters. The result is, however, that there is no joint at which a stress-diagram can be started immediately. The way out of the difficulty is, as in the case of the Fink truss, the previous determination of the stress in the horizontal tie by the method of sections.

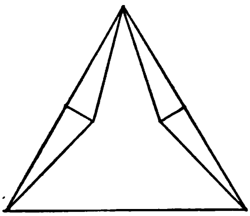


FIG. 31.

The Fink truss may, in fact, be regarded as a special case of this construction, in which the trussing of the top chord is more elaborate, and in which the horizontal tie and the lower chord of that trussing are coincident to some extent. Cf. Exercise 25.

Exercise 25. A triangular frame of 24 ft. span and 20 ft. rise is made with its top chord trussed as shown (Fig. 32), by means of struts, normal to them, 3 ft. 6 in. in length. The frame is supported by a hinge at the left end of the horizontal side and a set of rollers at the right end of the same bar. Loads P_1, P_2, P_3, P_4, and P_5 are applied at the joints and with the directions shown. Taking these forces at 2000, 10,000, 4000, 3000, and 4000 lbs. respectively construct the stress diagram.

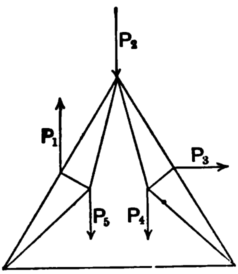


FIG. 32.

Suggestion. Here we can treat all external forces as acting outside the frame, by dotting such lines of action as may be needed and treating such parts of these lines as lie inside the frame periphery as if they were actual bars of the frame.

83. Counters.—Frames if made of a simple triangulation may be called upon to resist, in some of their members, both tension and compression according to variations in loading constantly occurring. Such reversal of stresses is to be avoided in the interests of economy and simplicity of design. If a diagonal in a given quadrilateral is replaced by its mate, if the loads remain unchanged, the stresses produced in the two

diagonals will be opposite in character. Hence, if in a quadrilateral where a diagonal might be called upon to resist compression, a mate to it, if present, would resist the stress in tension, and no compression could exist in either diagonal. A diagonal added for such a purpose is called a counter.

Counters in bridge trusses are idle when the truss carries a full symmetrical load. They are in action only under certain partial or unsymmetrical loads. Reversal of stress may be unpreventable in some members of certain kinds of trusses.

The greater the permanent load compared with the live, the less the likelihood of reversals, and the less there is for counters to do.

The presence of counters gives the frame an appearance of redundancy, but if the counters are of such a character or secured at the joints in such a way that they are incapable of resisting both tension and compression, the bars which must inevitably be out of action under the given loads can simply be ignored, and the analysis can proceed as usual.

Cf. Exercise 26 and Fig. 33.

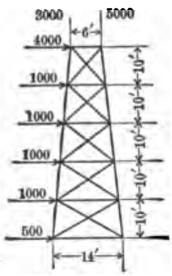


FIG. 33.

Exercise 26. A four-sided framed tower 40 ft. high and 14 ft. wide at the base, and whose other dimensions are as shown (Fig. 33), is made with a double set of slender diagonals throughout. In a gale each half of it is supposed to be subject to the forces shown. Assuming half the horizontal thrust taken up at each column base, construct the stress diagram.

Suggestion. In drawing the diagram of the frame dot the set of diagonals which are out of action. Cf. § 83.

84. Bent of a Mill Building.—A common and important type of structure subject to loads elsewhere than at the joints and hence exemplifying the partly framed, partly non-framed type of structure is the combination of columns with a truss constituting the bents of a mill-building and shown in Fig.

34. They are usually of steel throughout. The forces which give rise to their peculiarities are sidewise forces due to wind, tension of belts, shocks from traveling cranes, etc. The general relation between such structures and true frames is explained in §64 (*q.v.*), but it will be worth while to amplify that explanation by taking as an example the bent of Fig. 34

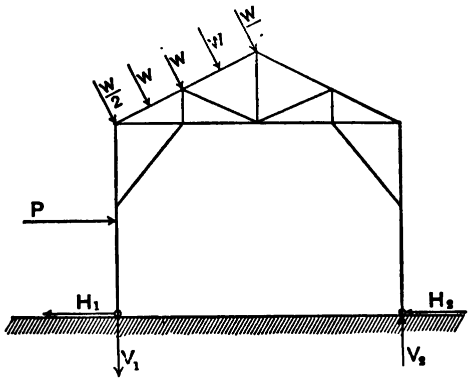


FIG. 34.

and discussing in detail the peculiar steps in its statical analysis.

The steps peculiar to the statical analysis of a mill-building bent are:

(*a*) The assumption of the points of application of the foundation reactions, R_1 and R_2.

(*b*) The analysis (by inspection) of the bent into its constituent true frame and flexural members.

(*c*) The determination of the forces brought to bear upon the joints of the frame by the flexural members.

The step (*a*) is one made necessary by the practical consideration that the columns are members of considerable breadth in the plane of the bent even at their bases. Moreover, they have to be anchored to the foundations so as to resist horizontal displacement as well as an actual lift. The result is an anchorage which offers considerable resistance to the rota-

tion of the column about its base. This resistance is of course a couple in the plane of the bent which combined with the requisite horizontal and vertical resistances amounts to the single forces R_1 and R_2 intersecting the axes of the columns at points at some distance y above their bases. The evaluation of y, which is dependent upon the evaluation of the couples, is a statically indeterminate problem. Consideration of the elasticity of the material * would lead to the assumption that y is half the distance between the column bases and the attachment of the knee-braces—the name given to the lowest of the bars secured directly to the column. This would minimize the stress in the columns, but would necessitate careful attention to the design and execution of the anchorages and foundations. In small structures in which such attention to the anchorage of the columns is not considered worth while, the existence of the couple may be ignored outright, and the structure treated as if y were zero and the bases of the columns hinged to the foundations. Then the couple actually materializing in the life of the structure is simply so much addition to the factor of safety. Cases might arise in which the designer would feel justified in assuming other values of y and proportioning the anchorages and other parts of the structure accordingly.

The points of application once decided upon, the distribution of the horizontal components between the two reactions (§ 80) would be made, the vertical components calculated (Case 2*b*) accordingly, and R_1 and R_2 established.

Step (*b*) is not difficult, for structures of the kind in question are simple triangulations with one or more bars subject to loads between joints, or extended beyond joints so as to receive loads on the extended part, usually at its end. The simple triangulation constitutes the true frame, and the bars just described

* Cf. Johnson, Bryan, and Turneaure's *Modern Framed Structures*, Art. 151.

are the flexural members,—the bars spoken of in §64 as doing double duty.

Step (*c*) consists of finding the reactions from the frame upon the flexural members necessary to hold them in place in opposition to the transverse loads to which they are subject. Reversing these reactions in sense, they become the forces transmitted to the joints of the frame by the flexural members.

The analysis of the frame then goes on as usual, establishing the tension and compression in all the members, including such as are the parts of the flexural members common to the frame. The flexural members have also shears and flexures, arising from their beam-action, to be determined, and when this is done the statical analysis of the structure is complete.

The whole process is one proceeding on a number of uncertainties, but there is no reason why it may not surely be kept on the side of safety and that, too, without serious lack of economy.

The bent of Fig. 34 is accordingly worked out as follows.

Assuming the columns hinged at their bases (step *a*), and the horizontal components H_1 and H_2 of the foundation reactions to be equal (§80), the vertical components, V_1 and V_2, of these reactions follow at once by Case 2*b*.

In Fig. 35 are shown (step *b*) the true frame, including the whole of the triangulation of bars in the bent, and, separate from it, the four flexural members,—the two top-chord bars subject to transverse loads, and the two columns from eaves to base.

The two top-chord bars will naturally be regarded (step *c*) as two centrally loaded beams requiring reactions of $\frac{1}{2}W$ from the frame joints at their ends and hence transmitting that amount to those joints. The two columns differ from the top-chord bars in the immaterial particular that their supporting joints are both on the same side of all the loads. V_1 and V_2

being regarded as transmitted directly up the column to the nearest joint of the true frame, the loads for the windward column are P and H_1, and for the leeward column H_2. The

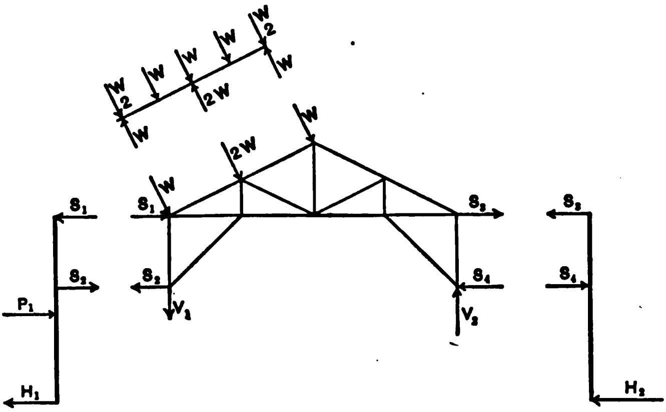


FIG. 35.

reactions S_1, S_2, S_3, and S_4 are determined by Case 2*b*. Reversing them in sense, they furnish the only remaining external forces in action on the true frame. The frame can now be analyzed as desired by familiar methods. Cf. Exercise 27.

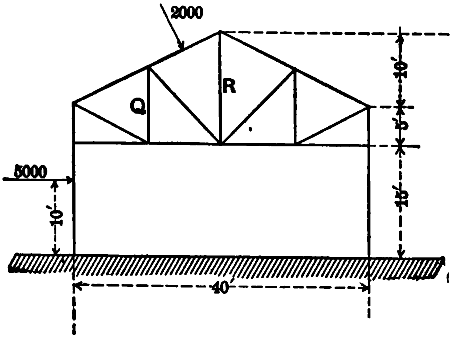


FIG. 36.

Exercise 27. A bent of a mill building is framed and subject to loads as shown in FIG. 36. The truss proper consists of four panels of equal length. The columns are to be assumed hinged at the base and the horizontal thrust divided equally between them. (Cf. §§ 80, 84.)

Construct the stress diagram and, preferably working with original frame as above shown, check algebraically the stresses in Q and R respectively, and record them side by side with the corresponding graphical results for comparison.

Suggestions. Determine the H's and V's algebraically. The determination of R will require the previous determination of one other bar. Show the bent completely analyzed into its constituent parts, giving the numerical magnitudes of all the external forces. (Cf. Fig. 35.)

The 2000-lb. force may be assumed to be normal to the top chord and applied midway between the two nearest joints. The knee-braces of this bent are seen to be horizontal.

85. Cantilever Bridge.—A cantilever bridge consists of one or more trusses or girders supported at one or both ends by ends of other trusses or girders which overhang their supports. The overhanging part is the cantilever whose prominence in this style of bridge gives it its name.

A cantilever bridge always requires more than two points of support, and the loads and reactions constituting usually a set of parallel forces, the reactions would seem at first glance to be indeterminate. Noticing, however, that the structure is composed of at least two or three structures, the reactions prove to be determinate. Cf. Exercise 28.

Exercise 28. A cantilever bridge proportioned and loaded as shown is supported by vertical reactions at A, B, C, and D. Determine these reactions graphically only.

Suggestions. Observe that the hinges at E and F divide the bridge

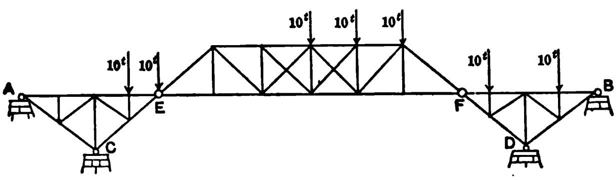


FIG. 37.

into three separate bodies, one resting upon two others, that none of these three bodies is in a statically indeterminate condition, and that therefore the whole structure is determinate.

Letter as usual around the figure and construct the string polygon for the given forces. One of the three strings then missing can be

located from the knowledge that the flexure at E and F must be zero (§ 70), and the other two follow at once.

86. Three-hinged Arch.—The structure typified in Fig. 38, consisting of two ribs (framed or non-framed, straight or curved, but usually curved) which are hinged together at their upper ends C and rest on hinges at their lower ends A and

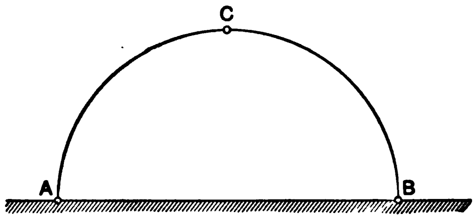


FIG. 38.

B, is called a three-hinged arch. It is used for roofs where the greatest spans are required, and for bridges as well.

It is a structure which evidently cannot stand up even under vertical loads without horizontal resistances at the supports. In this respect it is like an arch, and it easily lends itself to the pleasing curved form of an arch if desired. Moreover, it is the only statically determinate method of arch construction, and an important and fruitful subject of study accordingly.

There appears to be an indetermination in connection with the reactions at A and B. Here two forces are required of which only the points of application are given. Two magnitudes and two slopes must be determined, or else, if each reaction be replaced by two convenient components, four magnitudes.

Now, the hinge at C (assumed like the other hinges to be frictionless) cannot resist any force which does not pass through it, and this fact in connection with the three general conditions of equilibrium furnishes the basis for the necessary four equations and the problem is seen to be determinate.

These four conditions would be satisfied if two forces were

determined which, if applied at A and B respectively, would combine with all the forces between those points and C and yield two equal, opposite, and coincident resultants passing through C. These two forces would be the required reactions. Their intersection in the magnitude polygon would be the pole which (§ 54) would direct a string polygon for the given loads through the three given points A, B, and C in such a way that the strings through A and B would be the extreme strings of the whole given set, and the one through C the string common to the resultants of the two groups on each side of C.

Conversely, such a pole once determined, the two lines from it to the ends of the magnitude polygon for the given forces would give the magnitudes and directions required.

The methods of § 54 will therefore furnish the graphic and algebraic methods for the determination of the reactions.

Moreover, the string polygon through these three points for forces lettered in the order of their occurrence is the locus of the intersection (with any section) of the resultant of all the external forces on either side of the section. What is more, each string is the resultant of all the external forces on either side of it and its magnitude and sense can be found from the corresponding ray.

This string polygon once drawn, the stress in any bar under the most complicated system of loading can be determined by the method of sections (§ 73) with an equation of moments involving only four quantities, the known magnitude of the resultant, the required magnitude and their respective lever arms. These last can, of course, be scaled or calculated, but as a careful drawing is necessary in any event for the construction of the string polygon, scaling would commonly be far more convenient.

An alternative solution of the three-hinged arch is sometimes given which is based upon the fact that it may be

regarded as two separate structures, and the groups of loads coming on each assumed to act separately. Thus (Fig. 38) if the portion AC be unloaded while CB is loaded, the direction of the reaction at A must be through AC, and this reaction on CB and the one at B are determinate, falling under Case 3. Repeating the process for AC, two more partial reactions are determined. The resultants of the two reactions at each hinge are the reactions required for the structure.

It should be observed that the hinges A and B may be put at different levels without affecting the determinateness of the problem or its method of treatment.

The horizontal components at A and B may be furnished by abutments as with any arch, or, in cases where circumstances permit, by tie-rods connecting A and B, together with such anchorage as would be required with the given load to keep any truss from moving as a whole. With the tie-rod the structure becomes what might be regarded as a triangular truss, two of whose members are more or less curved and more or less frameworks themselves.

87. Line of Pressure.—The line of pressure in any structure is the locus of the intersections with successive section planes throughout the structure of the resultants of all the external forces on either side of those sections.

In case the loads are non-continuous, the line of pressure will be a broken line; with continuous loads, it is a curve which must be plotted point by point at sections taken at short intervals.

An example of the former case arises in connection with the three-hinged arch under a series of concentrated loads. The string polygon including the end reactions as extreme strings and passing through the three hinges is the line of pressure of that arch for the given loads, if the loads be lettered in the order of their occurrence.

An example of the latter case is a masonry dam exposed as it is to uninterrupted hydrostatic pressure. Points in the line of pressure are located in this case by taking successive horizontal sections and finding the resultants of the water pressure and weight of the masonry above that section. Where these resultants pierce the sections which define and limit their components are points in the line of pressure of the section of the dam.

Exercise 29. A three-hinged arch of 240 ft. span is subject to loads as shown in Fig. 39. The top chord is divided into eight equal bays, and

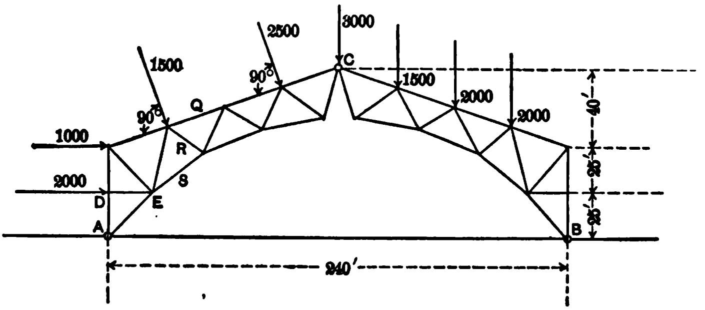


FIG. 39.

the bays of the bottom chord are each equal to AE in length, DE being horizontal. The joints of the bottom chords are on the arc of a circle of 145 ft. radius through A and B. Other dimensions as shown.

Required :

1. Reactions graphically by the method of § 54.
2. Reactions algebraically (§ 54), showing all lever arms dimensioned except such as are given directly by the main dimensions.
3. Line of pressure.
4. Stress diagram.

   Note that the stress diagrams of frames AC and BC will each have its own check.
5. Pressure on the hinge at C graphically.
6. Stress in the bars marked Q, R, and S with the aid of the line of pressure and scaled arms.
7. Stress in Q, R, and S from the stress diagram.
8. Tabulated record of results of 1, 2, 5, 6, and 7.

Suggestions. Use a large scale for the frame diagram, say 1 in. = 30 or 40 ft., and a moderate scale for the magnitude diagram.

Is the line of pressure a closed string polygon?

Exercise 30. Suppose the coordinates of the three hinges, *A*, *B*, and *C*, of a three-hinged arch to be expressed in feet as (0, 0), (60, 30), and (30, 50) respectively, and the ribs to be of any curvature and either framed or non-framed. Suppose a vertical load of 20 tons acting 20 ft. horizontally from *A*, and a horizontal force of 10 tons acting towards the right and applied between *B* and *C* and 10 ft. vertically above *B*. Determine the reactions algebraically.

Suggestion. The horizontal and vertical components of the two reactions would usually be a more convenient form for the expression of the required answer than the resultant reactions themselves.

Use the method of § 54, and check the results carefully.

88. Hammer-beam Truss.—Like all other roof trusses, the hammer-beam truss, Fig. 40, has vertical loads to resist

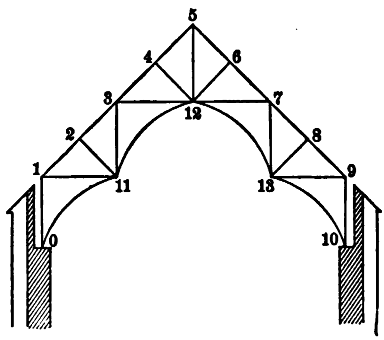


FIG. 40.

and also non-vertical ones. Unlike most trusses, some of its main members are curved. This curvature alone would not call for any special treatment here, for curved members may, for purposes of analysis, be considered replaced by straight ones connecting the same joints and the curvature left out of account till the design of the piece itself is taken up. The truss would then be a complete frame and would be treated like any other.

The combined effect of two circumstances, however, ren-

ders it necessary to approach the hammer-beam truss in a different way. These two circumstances are, (1) curved bars in frames at best work under very unfavorable conditions, and (2) in these trusses they are made of wood,—a material specially weak and difficult to connect. Moreover, even at best the shape of this truss is in itself unfavorable to rigidity. The result is that even under perfectly vertical loads the hammer-beam truss must be expected to produce thrust upon each supporting wall, as an arch would do, or as rafters without any collar-beam (or with the collar-beam too close to the ridge) would do. That means that the walls must be prepared to resist the spreading at the base of the truss which the curved ties are unable to prevent.

To estimate this thrust from vertical loads, a good way is to place no reliance whatever upon the upper curved members —to consider them absent. This reduces the truss to an incomplete frame—the parts below the joints 3 and 7 are mere framed inclined struts, reaching from their foothold on the wall to joints 3 and 7 of the truss 3, 4, 5, 6, 7, 12 which they support. The loads on this truss are known and hence the vertical load at the top of each strut. The vertical reaction at the base of each strut is, of course, equal and opposite, but as it is not coincident with it, the two form a couple, and the struts would fall into the building but for horizontal resistances at their upper ends along the bar 3, 12, 7. This horizontal resistance calls for an equal and opposite one at the base of each strut. These horizontal forces constitute the second couple which balances the first. The common magnitude of the horizontal forces may be taken as the thrust of the roof, and with the vertical reactions, which are easily found, there is no further obstacle to the construction of the stress diagram.

Finding thrusts as just described and making no claim on the upper curved members, and not an unduly severe one on

lower ones, leaves the former ready to act at their full value when the wind blows, and the latter in as favorable a condition as possible.

In determining wind stresses, the only thing to be done (except as described below) is to treat the truss as a complete frame, as explained above. The results thus obtained will be combined with those from the vertical loads.

The only difficulty, then, is to provide for the stresses. The stiffness of the rafters at 7 may have to be relied upon to assist the upper curved members, and the buttresses may have to rise to the level of 1 and 9 (as shown in the figure) in order to relieve the lower curved members.

This last would raise real points of support of the truss to 1 and 9, and 9, 10 and 0, 1 would become mere posts prevented from overturning by the buttresses, and 0, 11 and 10, 13 mere ornaments.

Of course in a large roof it may be necessary to rely upon the stiffness of the rafters, and even then higher stresses than would ordinarily be regarded as satisfactory may have to be tolerated. It may be advisable even to neglect all curved members and rely entirely on high buttresses and the stiffness of the rafters.

The hammer-Leam truss is a very imperfect structure as regards the economical application of material. The less the curvature of members the better from this point of view.

Exercise 31. With a vertical load of 1000 lbs. per bay of the top chord of a hammer-beam truss such as is shown in Fig. 40, determine the thrust on the walls on the assumption that the upper pair of curved bars is wholly inoperative, and construct the stress diagram. All bars of the truss are to be taken as horizontal, vertical, or inclined at 45° as shown, and 0, 1 and 9, 10 each of the same length as 1, 11 or 9, 13.

Suggestion. Draw a separate diagram of one of the framed struts 0, 1, 2, 3, 11, showing all the loads upon it, and determine the reactions upon it as an independent structure. The way is then clear for the stress diagram as usual.

89. Stresses Due to Moving Loads.—Some structures, as bridges and viaducts, are subject to loads varying greatly in magnitude and taking widely varying positions on the structure.

The special steps involved in the determination of the stresses in such a structure are

(1) The assumption of the maximum reasonable load or system of loads.

(2) The determination of the locations of this load or system of loads which will produce the extremes of stress in each and every part of the structure.

(3) The determination successively of the extremes of stress in each and every part of the structure due to loads assumed to be stationed at all the various points decided by the preceding step.

The first of these steps is an exercise of judgment in the light of experience of what the structure may reasonably be expected to carry, leaning of course towards too high rather than too low an estimate.

The second of the steps opens a large field in the study of special structures and systems of loading. In general it may be said to be a process based upon the observation of the stress in a given part of a structure due to a single force followed by a study of how and to what extent other forces may be grouped with it to reinforce its effect. The methods are a simple development of the general principles of statics as treated in Part I with the addition of various devices, largely graphical, for saving time and labor.

The third step is a straightforward solution of the simple statical problem of finding the stress in a certain part of any structure subject to a given set of loads.

The reader interested further in this subject will find it treated at length in Burr's *Stresses in Bridge and Roof*

Trusses, Johnson, Bryan, and Turneaure's *Modern Framed Structures*, Merriman and Jacoby's *Roofs and Bridges*, Parts I and II, Hoskins' *Graphic Statics*, DuBois' *Stresses in Framed Structures*, etc.

The suddenness of the application of the moving load or impact is not considered at all in this analysis. Allowance is made for that by certain empirical methods in connection with the proportioning of the parts to resist the extreme stresses determined as just described.

90. Stability of a Masonry Dam.—The lines of pressure for the various conditions to which a masonry dam is exposed afford a very important aid in the study of the stability and design of such a structure.

To construct such a line of pressure the cross-section of the dam is drawn to a large scale and divided into a suitable number of strips by a series of horizontal planes. The line of pressure will be located when there have been found the intersections with each of these planes of the resultant of all the forces acting upon the dam above that plane.

The gravity and hydrostatic forces on each strip are then determined (assuming the thickness of the strip normal to the paper to be one foot) and shown in their proper places.

This being done the successive resultants can be located by Case 1. The forces can sometimes be most conveniently lettered by beginning with the top strip and lettering the gravity forces, in the order of occurrence of their strips, ab, bc, etc., and lettering downward, giving the hydrostatic pressures in a similar order the letters ab', $b'c'$, etc.

A general string polygon will locate the successive resultants bb', cc', etc., and their intersections with the lower limits of the successive portions of the profile to which they belong will be points of the line of pressure.

An alternative lettering suitable to some cases would be to

letter the gravity and hydrostatic forces on the strips respectively *ab* and *bc*, *cd* and *de*, etc., beginning at the top. Taking the pole coincident with A, the strings *oc*, *oe*, etc., would be the required resultants.

Sometimes it may be more satisfactory to determine the locations of the resultants for the successive groups of gravity forces algebraically, and use the string polygon only for the hydrostatic forces. The intersections of the pairs of partial resultants will be points of the series of required resultants which can then be shown by transference from the magnitude diagram.

The location of the line of pressure in masonry arches differs from the preceding process only in the greater uncertainty about the elements of the external forces on the structure and the lack of a determinate point for beginning the line of pressure within the structure.

Exercise 32. A masonry dam with vertical upstream face has a cross-section determined by the following coordinates (origin at the top of the upstream face): (20,0), (20, − 17), (21, − 30), (31, − 50), (43, − 70), (58, − 90), (73, − 110), (0, − 110).

Required the lines of pressure (*a*) when the reservoir is empty, and (*b*) when the reservoir is full.

Weight of masonry to be taken at 150 lbs. per cu. ft. and of water at 62.5 lbs.

Statical work to be done graphically only.

91. Action and Reaction not Necessarily Normal to the Surface of the Contact of the Bodies.—The function of statics is to deal with the equilibrium of bodies without regard to the origin of the forces acting on them. In order, however, to deal with questions of equilibrium it is necessary to be familiar with the various possible sources of forces and the conditions under which they are effective in order correctly to prepare a problem for the application of purely statical methods.

In the exercises prescribed in the foregoing work, the

sources of all the forces dealt with might be classified broadly as either directly or indirectly the attraction of gravitation, which needs no further comment, and to an interaction between masses most naturally thought of as normal to their surface of contact. Accordingly special attention will now briefly be given to the possibility and causes of interactions between masses inclined to the normal to their surface of contact.

92. **Friction.**—It is a property of all bodies that they offer more or less resistance to being moved over one another. This resistance is a force in the plane of contact always to be considered when one body rests upon another and the other forces acting on the body have an unbalanced component parallel to that plane. This kind of resistance is called **friction**. The resultant of the friction with the normal resistance is a force acting on the body inclined to the normal to their surface of contact.

The elements of a force due to friction have the following characteristics.

The point of application may always be taken in the surface of contact.

The direction is always opposite to any unbalanced component of the other forces parallel to the surface of contact.

The magnitude, like that of any other passive resistance, is variable. It is as large as it has to be (up to a certain limit) to maintain equilibrium and no larger. In this respect, friction, or tangential resistance, is precisely like the normal resistance. The limit in the latter case is the ultimate compressive strength of the weaker of the two bodies and, in the former, the limit is a certain percentage (called the **coefficient of friction**) of the normal pressure existing at the time in question. If the former be overstepped penetration or crushing will occur, if the latter, sliding or rolling. Both limits are experimentally

determined for various materials and conditions, and data regarding them can be found on record in the standard reference books.

The coefficient of friction is commonly designated by the letter μ. Its values vary greatly according to the materials in contact, the smoothness of the surface of contact, the degree of lubrication, whether the motion is likely to be sliding or rolling.

When the limit of friction is reached and the body is about to move, the reaction upon this body from the one under it is at a limiting inclination to the normal to the surface of contact called the **angle of friction.** The tangent of this limiting angle is evidently the ratio of the limiting frictional resistance to the normal pressure with which it is associated. That is, **the coefficient of friction is the tangent of the angle of friction.**

If greater resistance to motion on the surface of contact be required than friction can be trusted to furnish, recourse can be had to various devices, prominent among them the use of one or more bodies penetrating each of the two given ones, passing through their surface of contact and furnishing the required resistance in that surface by virtue of their shearing strength. Nails, bolts, rivets, and dowels are examples of the bodies used for such purpose.

What is of most importance with regard to friction from the point of view of statics is that it is a source of passive force tangential to a surface of contact between two bodies just as normal resistance is a source of passive force normal to that surface. Of course a force due to friction is treated statically just like any other force.

Example 1. A body of weight W rests upon a rough plane inclined α to the horizon and is subject to a horizontal force P. The coefficient of friction being μ, between what two limits may P vary while the body remains at rest?

Solution. The forces acting on the body are the active forces P and W, and the passive ones N and $T = \mu N$. When T acts downward as shown, Fig. 41, P is at its upper limiting value and the body is about to move up the plane.

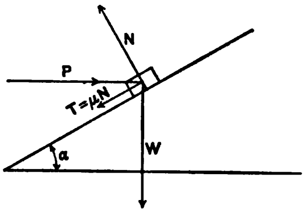


FIG. 41.

Statically the problem is one to determine two magnitudes in a set of concurrent forces whose lines of action are all given (Case $2a$). The unknowns are P and N, T being known as soon as N is. The regular solution of Case $2a$ can now be applied. No numerical data being given the algebraic method will be used.

Resolving the four forces along and at right angles to the plane for the A_x and A_y equations there results

$$P \cos \alpha - \mu N - W \sin \alpha = 0,$$

$$P \sin \alpha - N + W \cos \alpha = 0.$$

Eliminating N, it appears that

$$P = \frac{\sin \alpha + \mu \cos \alpha}{\cos \alpha - \mu \sin \alpha} W.$$

Reversing the direction of T the other limiting value of P is found by similar means to be

$$P = \frac{\sin \alpha - \mu \cos \alpha}{\cos \alpha + \mu \sin \alpha} W.$$

The same result might have been reached by reversing the sign of μ in the other value of P.

Example 2. A uniform ladder rests upon a horizontal floor and against a vertical wall. The floor and wall are of different materials and the coefficients of friction between them and the material of the ladder are respectively .30 and .20. What is the least inclination to the horizon at which the ladder can rest?

Solution. The ladder is to be in equilibrium under three forces: W its weight acting at its center of gravity, P and Q at its upper and lower ends respectively with directions inclined to the normals to the wall and floor by their respective angles of friction, and in such a way as to give P an upward component and Q a component towards the wall. The part that pure statics plays in this problem is to furnish the condition that W must pass through the intersection of P and Q. To this end the middle of the ladder must be in the same vertical with the intersection of P and Q. The inclination, ϕ, of the ladder when this condition is satisfied will be the limiting inclination required.

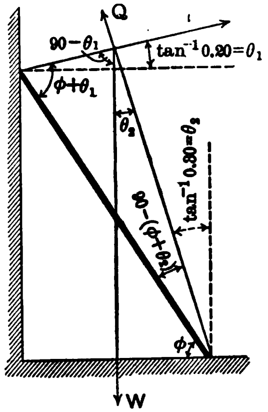


FIG. 42.

The determination of this inclination is henceforth purely a geometric and trigonometric process. Thus can be written (Fig. 42),

$$\frac{\cos(\phi + \theta_2)}{\sin \theta_2} = \frac{\sin(\phi + \theta_1)}{\cos \theta_1},$$

whence can be established $\tan \phi = \dfrac{\text{ctn } \theta_2 - \tan \theta_1}{2}$, and θ_2 and θ_1 being given respectively as $\tan^{-1} 0.30$ and $\tan^{-1} 0.20$, it follows that

$$\phi = \tan^{-1} 1.567 = 57^\circ 27'. \qquad \textit{Ans.}$$

Additional problems involving friction may be found in abundance in such works as those mentioned at the end of Chapter VI.

APPENDIX.

ADDITIONAL REGARDING THE SCOPE OF PURE STATICS.

In this Appendix, as in Chapter V, the term Pure Statics is used to designate the treatment of problems in which the data consist exclusively of elements of forces, the desiderata include only other elements of forces, and the conditions of equilibrium of rigid bodies form the only determining conditions.

A somewhat different class of statical problems is sometimes met, especially in text-books, in which the object is to determine the position in which a given body will rest under given forces and with given conditions of support. The desiderata here are geometrical relations between bodies consistent with equilibrium, and not elements of forces. Such problems might not improperly be called geometrico-statical problems to distinguish them from purely statical problems. An illustration of such is to be found in Example 2 of §92. Their solution consists in using the conditions of equilibrium to identify the position of the body for which the forces acting would be interrelated in a manner consistent with equilibrium, and then in using sufficient geometric insight to derive a simple definition of that position,—the latter of the two steps often proving the more puzzling of the two.

Moreover, as stated in §46, a large part and often the most difficult part of an ordinary statical problem in practice is the discernment of the data in proper shape for insertion into the purely routine processes of pure statics.

Though pure statics must accordingly be understood to have a restricted meaning, nevertheless within its domain will be found nearly all of the statical problems of engineering. The mathematical extent of this domain offers an inviting field of inquiry with a view to finding out how many and what problems might arise, which would not be obviously beyond its purview by involving more unknowns than there are determining conditions.

As a simple matter of permutations and combinations an early limit is set to the number of such problems. It is merely a question of the number of ways in which groups of elements, each group containing as many elements as there are determining equations, can be selected without repetition from the total number of elements which pertain to forces not exceeding in number the determining equations. It will be convenient to treat forces in two groups, (*a*) non-concurrent (excluding parallel) forces and (*b*) concurrent (including parallel) forces, in which the determining equations are three and two respectively, and the limiting number of combinations eighty-four and fifteen as deduced in §44. As implied in §44, many of these combinations are, after all, statical identities. For example, a magnitude, direction, and point of application can be selected from a group containing three each of these elements in three different ways, but in each case the statical significance is the same, viz., that the magnitude, direction, and point of application of one of a group of forces are to be evaluated. Collecting the statical identities as separate cases, there are found for non-concurrent forces only twenty such cases, numbered in the following table I–XX, and for concurrent only nine, numbered XXI–XXIX.

It should be borne in mind that for non-concurrence, as here understood, it is necessary and sufficient that the resultant of all the known forces shall not include a point common to

three unknowns nor be parallel to them, and for concurrence that this resultant shall intersect the point common to two unknowns or be parallel to them. It hardly need be added that with non-concurrent forces if three unknowns should have a point in common or be parallel, the problem would always be incapable of solution.

The following table gives a list of all the twenty-nine cases, with the number of statical identities included in each case. P, α, and m are used to designate magnitude, direction, and point of application respectively of any of three forces, Q, R, and S, which may involve unknowns. The sense may be understood to be unknown whenever either the magnitude or the direction is unknown.

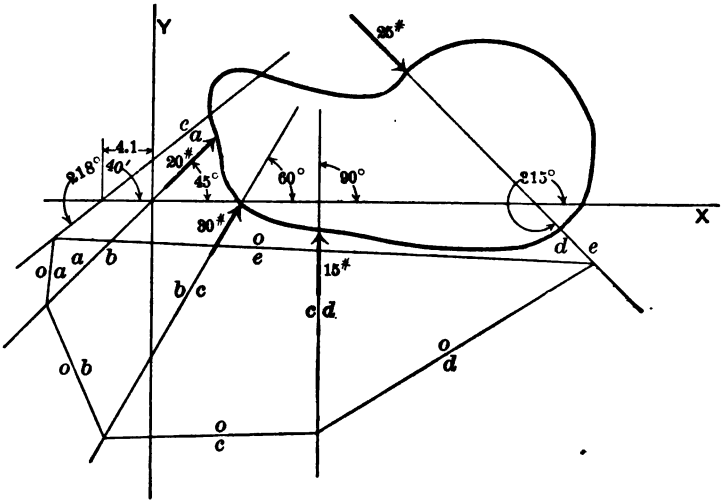
The reader who has mastered the four cases of §45 will find the additional ones an interesting field for further practice with the same methods, adjusting them of course to the peculiarities of each case. The graphical method is recommended as especially convenient for this investigation. The problem in the first twenty cases is then uniformly to close a magnitude polygon and a string polygon, observing that in the last nine cases, insuring the concurrence of the forces may replace the closure of the string polygon. The number of forces involved need never exceed four, any number of given forces being considered to be represented by any single force as their resultant. Some of these new cases will be found incapable of solution, or capable of one or more real solutions, according to the relations between the data in each case. Others are always indeterminate. Very few are always capable of only a single solution.

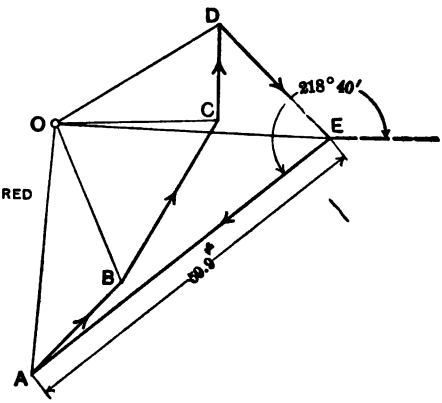
Some of the last nine cases might be regarded as special cases already included in the first twenty.

No.	Unknown Elements Pertaining to									No. of Statical Identities Included.	Remarks.	
	Q			*R*			*S*					
I	*P*	*α*	*m*							3	Case 1 of § 45	Non-concurrent (excluding parallel) forces.
II	*P*			*P*	*α*					6	Case 3 of § 45	
III	*P*			*P*		*m*				6		
IV		*α*			*α*	*m*				6		
V		*α*		*P*	*α*					6		
VI			*m*	*P*		*m*				6		
VII			*m*		*α*	*m*				6		
VIII			*m*	*P*	*α*					6		
IX		*α*		*P*		*m*				6		
X	*P*				*α*	*m*				6		
XI	*P*			*P*			*P*			1	Case 4 of §45	
XII	*P*			*P*				*α*		3		
XIII	*P*			*P*					*m*	3		
XIV		*α*			*α*			*α*		1		
XV		*α*			*α*		*P*			3		
XVI		*α*			*α*				*m*	3		
XVII			*m*			*m*			*m*	1		
XVIII			*m*			*m*	*P*			3		
XIX			*m*			*m*		*α*		3		
XX	*P*				*α*				*m*	6		
XXI	*P*		*α*							2		Concurrent (including parallel) forces.
XXII	*P*		*m*							2		
XXIII		*α*	*m*							2		
XXIV	*P*				*α*					2		
XXV	*P*					*m*				2		
XXVI		*α*				*m*				2		
XXVII	*P*			*P*						1	Case 2 of § 45	
XXVIII		*α*			*α*					1		
XXIX			*m*			*m*				1		

PROBLEM.—Determine the EQUILIBRANT of the following set of forces: (20 ll
What is the RESULTANT of the set?

GRAPHIC SOLUTION





E A IS THE REQUIRED EQUILIBRANT.

Scales { 1 in. = 15 units of length
1 in. = 30 lbs.

SE 1

bs. 45° 0,0), (15 lbs. 90° 13,0), (30 lbs. 60° 7,0), and (25 lbs. 315° 30,0).

ALGEBRAIC SOLUTION

Let (E a x,y) be the force sought, then

For no translation:

$$A_x: E\ \cos a + 20 \overset{.707}{\underset{+14.14}{\cos}} 45° + 15 \overset{0}{\underset{0}{\cos}} 90° + 30 \overset{.500}{\underset{+15.00}{\cos}} 60° + 25 \overset{.707}{\underset{+17.68}{\cos}} 315° = 0$$

$$E\ \cos a = -46.82$$

$$A_y: E\ \sin a + 20 \overset{.707}{\underset{+14.14}{\sin}} 45° + 15 \overset{1.00}{\underset{+15.00}{\sin}} 90° + 30 \overset{.866}{\underset{+25.98}{\sin}} 60° + 25 \overset{-.707}{\underset{-17.68}{\sin}} 315° = 0$$

$$E\ \sin a = -37.44$$

$$\tan a = \frac{E\ \sin a}{E\ \cos a} = \frac{-37.44}{-46.82} = .800$$

$$a = 218°\ 40'$$

$$E = \frac{E\ \sin a}{\sin a} = \frac{-37.44}{-.625} = 59.9$$

For no rotation:

Taking a point on the X-axis for the point of application of E, y becomes zero and x is determined as follows:

$$B: E \overset{0}{\underset{0}{\cos}} a \times y - \overset{-37.44}{\underset{-37.44x}{E \sin a}} \times x + 20 \overset{}{\underset{0}{\cos}} 45° \times 0 - 20 \underset{0}{\sin} 45° \times 0 + 15 \underset{0}{\cos} 90° \times 0$$

$$-15 \overset{15.00}{\underset{-195.00}{\sin}} 90° \times 13 + 30 \underset{0}{\cos} 60° \times 0 - 30 \overset{25.98}{\underset{-181.86}{\sin}} 60° \times 7 + 25 \underset{0}{\cos} 315° \times 0 - 25 \overset{-17.68}{\underset{+530.40}{\sin}} 315° \times 30 = 0$$

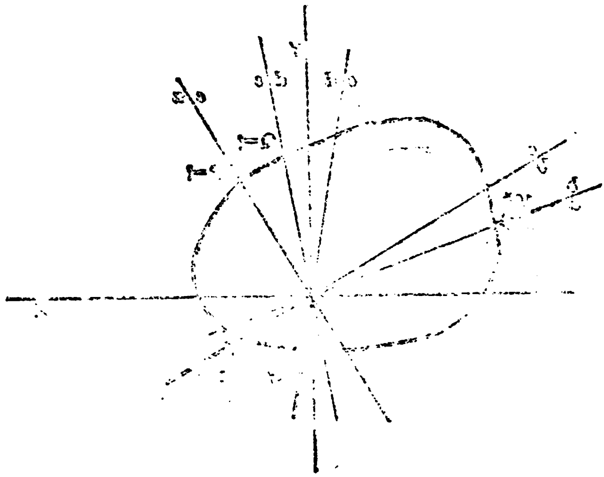
$$x = \frac{-153.54}{37.44} = -4.10$$

RESULTS

Algebraic		Graphic	
Equilibrant	(59.9 lbs. 218° 40′ −4.10,0)	Equilibrant	(59.9 lbs. 218° 40′ −4.1,0)
Resultant	(59.9 lbs. 38° 40′ −4.10,0)	Resultant	(59.9 lbs 38° 40′ −4.1,0)

PROBLEM—In the following set of forces in equilibrium, (as fig. ...) determine the unknowns.

GRAPHIC SOLUTION

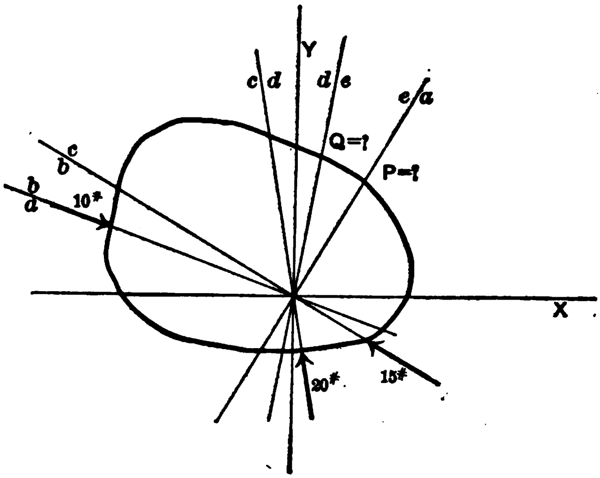


CAS

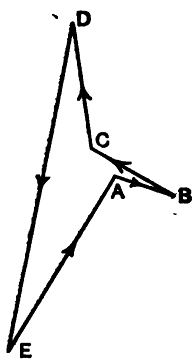
PROBLEM.—In the following set of forces in equilibrium, (20 lbs. 100° 0,0

determine the unknowns.

GRAPHIC SOLUTION



NO COUPLE BEING POSSIBLE, THE STRING POLYGON IS SUPERFLUOUS



Q–DE P–EA

Scale 1 in. = 15 lbs.

CE 2a

…, (15 lbs. 150° 0,0), (P = ? $\begin{matrix}60^\circ\\ \text{or}\\ 240^\circ\end{matrix}$ 0,0), (10 lbs. 340° 0,0), and (Q = ? $\begin{matrix}80^\circ\\ \text{or}\\ 260^\circ\end{matrix}$ 0,0),

ALGEBRAIC SOLUTION

$$A_x:\ 20\underset{-3.48}{\overset{-.174}{\cos}}\,100^\circ+15\underset{-12.99}{\overset{-.866}{\cos}}\,150^\circ+P\underset{+0.500P}{\overset{.500}{\cos}}\,60^\circ+10\underset{+9.40}{\overset{.940}{\cos}}\,340^\circ+Q\underset{+0.174Q}{\overset{.174}{\cos}}\,80^\circ=0$$

$$.500\,P+.174\,Q=7.07$$

$$A_y:\ 20\underset{+19.70}{\overset{.985}{\sin}}\,100^\circ+15\underset{+7.50}{\overset{.500}{\sin}}\,150^\circ+P\underset{+0.866P}{\overset{.866}{\sin}}\,60^\circ+10\underset{-3.42}{\overset{-.342}{\sin}}\,340^\circ+Q\underset{+0.985Q}{\overset{.985}{\sin}}\,80^\circ=0$$

$$.866\,P+.985\,Q=23.78$$

$$P=\ \ 32.2=(32.2\quad 60^\circ)$$

$$Q=-52.6=(52.6\quad 260^\circ)$$

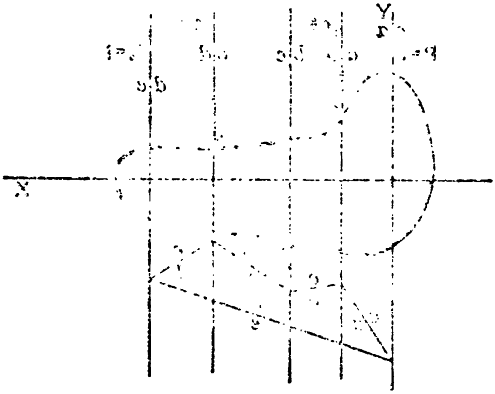
B: No couple being possible, the equation of moments is superfluous

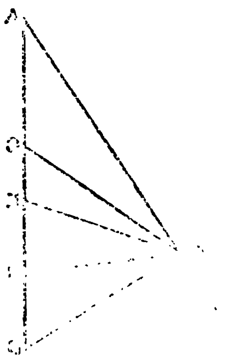
RESULTS

Algebraic	Graphic
$P=(32.2$ lbs. $60^\circ)$	$P=(32.3$ lbs. $60^\circ)$
$Q=(52.6$ lbs. $260^\circ)$	$Q=(52.5$ lbs. $260^\circ)$

PROBLEM — In the following ... determine the unknowns.

GRAPHIC SOLUTION



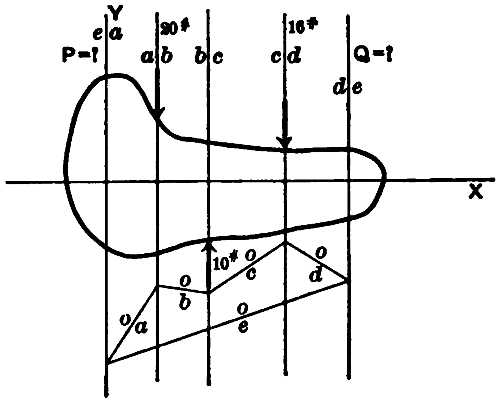


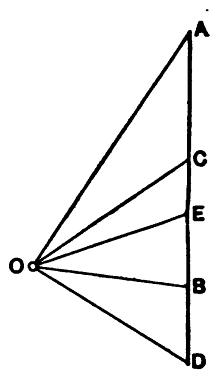
Scales { 1 in. = ... ft.
1 in. = ... lbs

CA

PROBLEM —In the following set of forces in equilibrium, (20 lbs. 270° 4,0),
determine the unknowns.

GRAPHIC SOLUTION





Scales $\left\{ \begin{array}{l} \text{1 in.} = \text{15 units of length} \\ \text{1 in.} = \text{15 lbs} \end{array} \right.$

P=EA Q=DE

E 2b.

$P = ?\ \begin{matrix}90^\circ\\ \text{or}\\ 270^\circ\end{matrix}\ 0{,}0)$, (10 lbs. 90° 8,0), (16 lbs. 270° 14,0), and $(Q = ?\ \begin{matrix}90^\circ\\ \text{or}\\ 270^\circ\end{matrix}\ 19{,}0)$,

ALGEBRAIC SOLUTION

Solution by A_x, A_y, and B.

$$A_x:\ 20 \overset{0}{\underset{0}{\cos}} 270^\circ + P \overset{0}{\underset{0}{\cos}} 90^\circ + 10 \overset{0}{\underset{0}{\cos}} 90^\circ + 16 \overset{0}{\underset{0}{\cos}} 270^\circ + Q \overset{0}{\underset{0}{\cos}} 90^\circ = 0$$

$$A_y:\ 20 \overset{-1.0}{\underset{-20.0}{\sin}} 270^\circ + P \overset{1.0}{\underset{+P}{\sin}} 90^\circ + 10 \overset{1.0}{\underset{+10.0}{\sin}} 90^\circ + 16 \overset{-1.0}{\underset{-16.0}{\sin}} 270^\circ + Q \overset{1.0}{\underset{+Q}{\sin}} 90^\circ = 0$$

$$P + Q = 26.0$$

$$B:\ 20 \underset{0}{\cos 270^\circ \times 0} - 20 \overset{-20.0}{\underset{+80}{\sin}} 270^\circ \times 4 + P \underset{0}{\cos 90^\circ \times 0} - P \underset{0}{\sin 90^\circ \times 0} + 10 \underset{0}{\cos 90^\circ 0}$$

$$- 10 \overset{10.0}{\underset{-80.0}{\sin}} 90^\circ \times 8 + 16 \underset{0}{\cos 270^\circ \times 0} - 16 \overset{-16.0}{\underset{+224.0}{\sin}} 270^\circ \times 14 + Q \underset{0}{\cos 90^\circ \times 0} - Q \overset{1.0}{\underset{-19Q}{\sin}} 90^\circ \times 19 = 0$$

$$-19Q = -224.0$$

$$Q = 11.8$$

By A_y,

$$P = 26.0 - Q = 26.0 - 11.8$$

$$P = 14.2$$

Alternative form of the preceding using moments alone.

Center of moments at (0,0)

$$P \times 0 + \overset{80}{20 \times 4} - \overset{-80}{10 \times 8} + \overset{224}{16 \times 14} + Q \times 19 = 0$$

$$Q = \frac{-224}{19} = -11.8 = (11.8\ \ 90^\circ)$$

Center of moments at (19,0)

$$P \times 19 - \overset{-300}{20 \times 15} + \overset{110}{10 \times 11} - \overset{-80}{16 \times 5} + Q \times 0 = 0$$

$$P = \frac{270}{19} = 14.2 = (14.2\ \ 90^\circ)$$

Check by A_y:

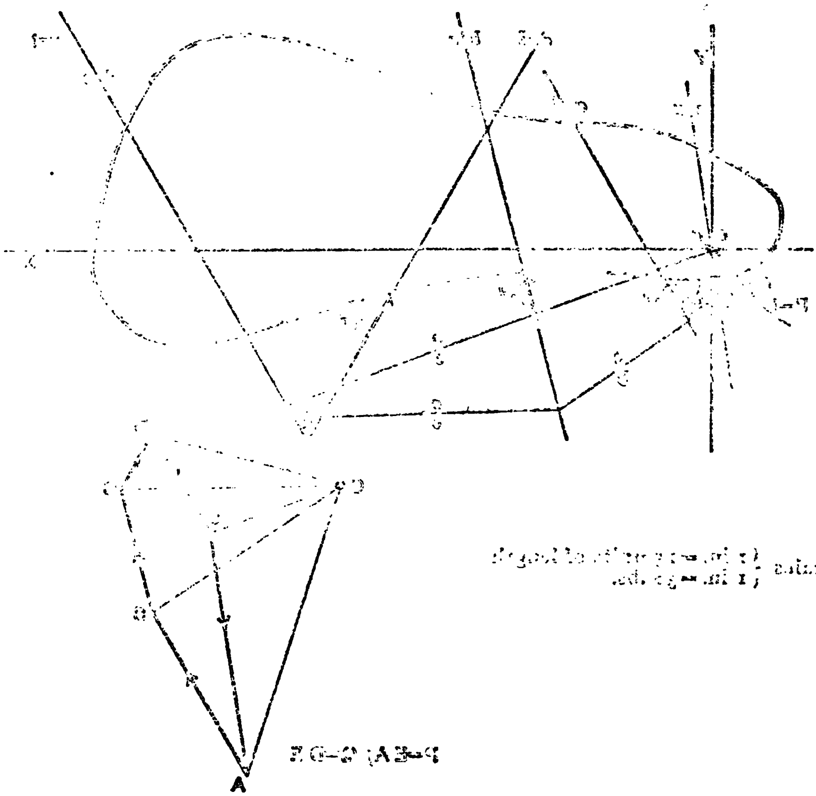
$$14.2 - 20 + 10 - 16 + 11.8 = 0.$$

RESULTS

Algebraic	Graphic
$P = (14.2$ lbs $90^\circ)$	$P = (14.2$ lbs. $90^\circ)$
$Q = (11.8$ lbs $90^\circ)$	$P = (11.8$ lbs. $90^\circ)$

mine the unknowns.

GRAPHIC SOLUTION.

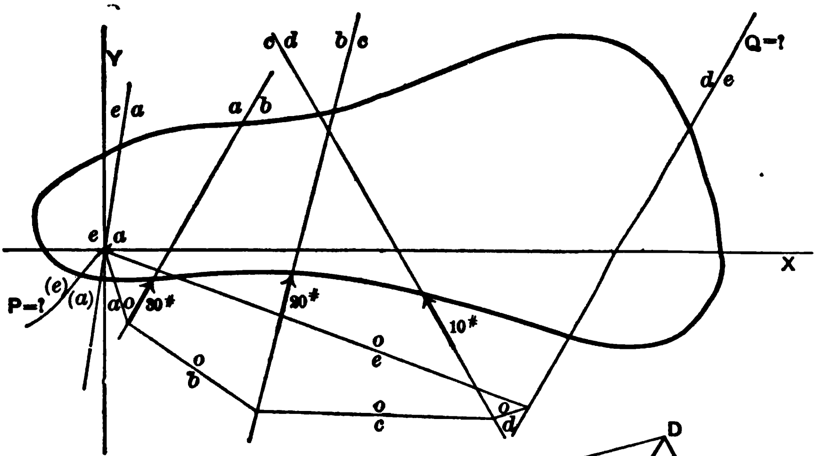


Scales { 1 in. = ... ft. of length ; 1 in. = ... lbs.

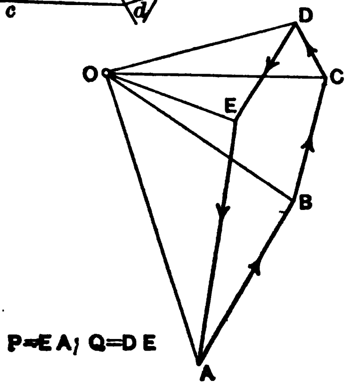
P=EA; Q=G-F

PROBLEM.—In the following set of forces in equilibrium, ($P=?$ $a=?$ o,o), mine the unknowns.

GRAPHIC SOLUTION.



Scales $\begin{cases}\text{1 in.} = \text{15 units of length}\\ \text{1 in.} = \text{30 lbs.}\end{cases}$



3.

lbs. 120° 23,0), ($Q = ?$ $\begin{matrix}60^\circ\\ \text{or}\\ 240^\circ\end{matrix}$ 40,0), (20 lbs. 75° 15, 0), and (30 lbs. 60° 5,0) deter-

ALGEBRAIC SOLUTION

Center of moments at the point of application of the force which is otherwise unknown, which, in this problem, is the point (0,0),

$$\boldsymbol{B}:\quad P\underset{0}{\cos a}\times 0 - P\underset{0}{\sin a}\times 0 + 10\underset{0}{\cos 120^\circ\times 0} - 10\underset{-199.18}{\overset{.866}{\sin}\,120^\circ\times 23} + Q\underset{0}{\cos 60^\circ\times 0}$$

$$-\underset{-34.64\,Q}{Q\overset{.866}{\sin}\,60^\circ\times 40} + 20\underset{0}{\cos 75^\circ\times 0} - 20\underset{-289.80}{\overset{.966}{\sin}\,75^\circ\times 15} + 30\underset{0}{\cos 60^\circ\times 0} - 30\underset{-129.90}{\sin 60^\circ\times 5} = 0$$

$$-34.64\,Q = 618.88$$

$$Q = -17.9 = (17.9 \quad 240^\circ)$$

Using this value of Q in the next two equations,

$$\boldsymbol{A_x}:\quad P\cos a + \underset{-5.00}{10\overset{-.500}{\cos}\,120^\circ} + \underset{-8.85}{\overset{17.9}{Q}\overset{-.500}{\cos}\,240^\circ} + \underset{+5.18}{20\overset{.259}{\cos}\,75^\circ} + \underset{+15.00}{30\overset{.500}{\cos}\,60^\circ} = 0$$

$$P\cos a = -6.23$$

$$\boldsymbol{A_y}:\quad P\sin a + \underset{+8.66}{10\overset{.866}{\sin}\,120^\circ} + \underset{-15.50}{\overset{17.9}{Q}\overset{-.866}{\sin}\,240^\circ} + \underset{+19.32}{20\overset{.966}{\sin}\,75^\circ} + \underset{+25.98}{30\overset{.866}{\sin}\,60^\circ} = 0$$

$$P\sin a - = 38.46$$

$$\tan a = \frac{P\sin a}{P\cos a} = \frac{-38.46}{-6.23} = 6.17$$

$$a = 260^\circ\ 48'$$

$$P = \frac{P\cos a}{\cos a} = \frac{-6.23}{-.160} = 38.9$$

RESULTS

Algebraic		Graphic	
$P = 38.9$ lbs.	260° 48′	$P = 38.9$ lbs.	261°
$Q = 17.9$ lbs.	240°	$Q = 17.9$ lbs.	240°

[illegible]

RESULTS

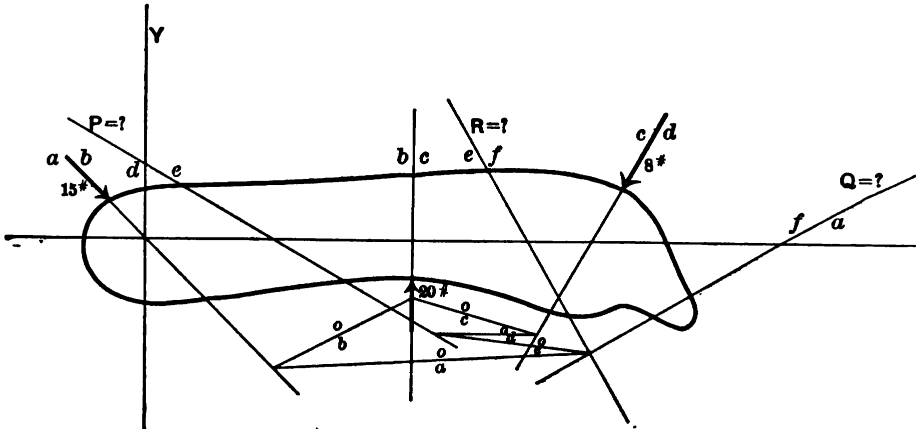
Algebraic	Graphic
P = [illegible] lbs. [illegible]° 48'	P = [illegible] lbs. 261°
Q = 17.9 lbs. 240°	Q = 17.9 lbs. 240°

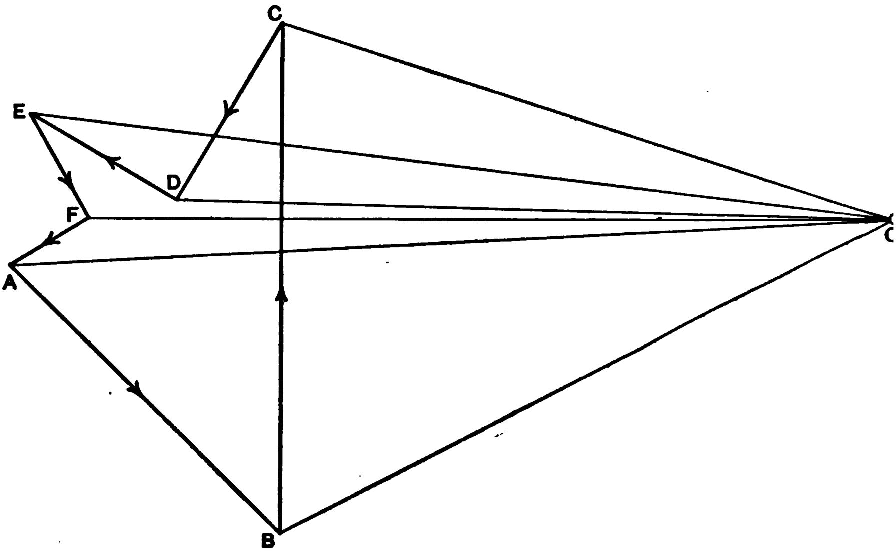
C

PROBLEM.—In the following set of forces in equilibrium, (15 lbs.

and $(R \overset{120^\circ}{\underset{300^\circ}{\text{or}}} 30,0)$, determine P, Q, and R.

GRAPHIC SOLUTION





P–DE Q–FA R–EF

Scales { 1 in. = 15 units of length
1 in. = 7.5 lbs.

SE 4.

$15^\circ\ 0{,}0),\ (P\ \begin{matrix}150^\circ\\ \text{or}\\ 330^\circ\end{matrix}\ 10{,}0),\ (20 \text{ lbs. } 90^\circ\ 21{,}0),\ Q\ \begin{matrix}30^\circ\\ \text{or}\\ 210^\circ\end{matrix}\ 50{,}0),\ (8 \text{ lbs. } 240^\circ\ 35{,}0),$

ALGEBRAIC SOLUTION

Solution by A_x, A_y, and B.

$$A_x:\ 15 \underset{10.61}{\overset{.707}{\cos 315^\circ}} + P \underset{-0.866P}{\overset{-.866}{\cos 150^\circ}} + 20 \underset{0}{\overset{0}{\cos 90^\circ}} + Q \underset{+0.866Q}{\overset{.866}{\cos 30^\circ}} + 8 \underset{-4.00}{\overset{-.500}{\cos 240^\circ}} + R \underset{-0.50R}{\overset{-.500}{\cos 120^\circ}} = 0$$

$$-0.866P + 0.866Q - 0.50R = -6.61$$

$$A_y:\ 15 \underset{-10.61}{\overset{-.707}{\sin 315^\circ}} + P \underset{+0.5P}{\overset{.500}{\sin 150^\circ}} + 20 \underset{+20.00}{\overset{1.00}{\sin 90^\circ}} + Q \underset{+0.50Q}{\overset{.500}{\sin 30^\circ}} + 8 \underset{-6.93}{\overset{-.866}{\sin 240^\circ}} + R \underset{+0.866R}{\overset{.866}{\sin 120^\circ}} = 0$$

$$0.50P + 0.50Q + 0.866R = -2.46$$

Center of Moments at (0,0)

$$B:\ 15 \underset{0}{\cos 315^\circ \times 0} - 15 \underset{0}{\sin 315^\circ \times 0} + P \underset{0}{\cos 150^\circ \times 0} - P \underset{+50.P}{\overset{0.50P}{\sin 150^\circ}} \times 10 + 20 \underset{0}{\cos 90^\circ \times 0}$$

$$-20 \underset{-420.00}{\overset{20.00}{\sin 90^\circ \times 21}} + Q \underset{0}{\cos 30^\circ \times 0} - Q \underset{-25.00Q}{\overset{0.50Q}{\sin 30^\circ}} \times 50 + 8 \underset{0}{\cos 240^\circ \times 0} - 8 \underset{+242.55}{\overset{-6.93}{\sin 240^\circ \times 35}}$$

$$+R \underset{0}{\cos 120^\circ \times 0} - R \underset{-25.98R}{\overset{.866R}{\sin 120^\circ}} \times 30 = 0$$

$$-5.00P - 25.00Q - 25.98R = 177.45$$

Combining the three equations, A_x, A_y, and B it appears that

$P = +6.82 = (6.82\ 150^\circ)$; $Q = -3.54 = (3.54\ 210^\circ)$; $R = -4.73 = (4.73\ 300^\circ)$

Alternative form of the preceding, using moments alone:

To find P, take center of moments at (x_1, y_1), the intersection of Q and R. Then since

$$y - .577x = -28.87 \quad \text{and} \quad y + 1.732x = 51.96$$

are the equations of the lines of action of Q and R respectively, it appears that

$$x_1 = 35.0 \quad \text{and} \quad y_1 = -8.66.$$

Center of Moments at $(35.0, -8.66)$, transforming co-ordinates:

$$15(8.66 \underset{+6.12}{\overset{.707}{\cos 315^\circ}} - -35 \underset{-24.75}{\overset{-.707}{\sin 315^\circ}}) + P(8.66 \underset{-7.50}{\overset{-.866}{\cos 150^\circ}} - -25 \underset{+12.50}{\overset{.500}{\sin 150^\circ}})$$

$$+20(8.66 \underset{0}{\overset{0}{\cos 90^\circ}} - -14 \underset{14.00}{\overset{1.00}{\sin 90^\circ}}) + 8(8.66 \underset{-4.33}{\overset{-.500}{\cos 240^\circ}} - 0 \underset{0}{\overset{-.866}{\sin 240^\circ}}) + Q \times 0$$

$$+R \times 0 = 0$$

$5.0P = 279.45 - 280.00 + 34.64 = 34.09$. Whence $P = +6.82 = (6.82\ 150^\circ)$

To find Q, take center of moments at intersection of P and R and proceed as for P.

Similarly, to find R, take center of moments at the intersection P and Q.

In practical cases the algebraic work usually proves much simpler than for the general case used in this Plate. See Plate V.

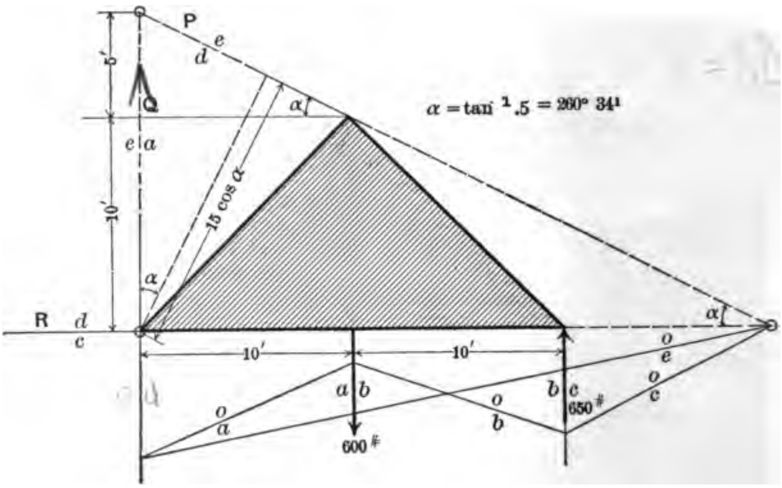
RESULTS

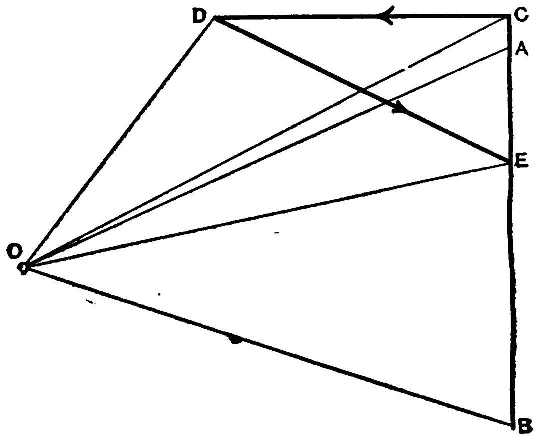
Algebraic	Graphic
$P = (6.82$ lbs. $150^\circ)$	$P = (6.8$ lbs. $150^\circ)$
$Q = (3.54$ lbs. $210^\circ)$	$Q = (3.6$ lbs. $210^\circ)$
$R = (4.73$ lbs. $300^\circ)$	$R = (4.7$ lbs. $300^\circ)$

CASE 4 (secon

PROBLEM.—A body in the shape of an isosceles triangle, base 20 feet, altitude 1
are known in line of action only. Determine P, Q, and R.

GRAPHIC SOLUTION





P-DE Q-EA R-CD

Scales $\left\{ \begin{array}{l} 1 \text{ in.} = 9 \text{ ft.} \\ 1 \text{ in.} = 300 \text{ lbs.} \end{array} \right.$

second example).

feet, is in equilibrium under the action of the forces shown; of these P, Q, and R

ALGEBRAIC SOLUTION

Center of moments at intersection of P and Q

$R \times 15 + 600 \times 10 - 650 \times 20 + P \times 0 + Q \times 0 = 0$

$$R = -\frac{\overset{6000}{600 \times 10} - \overset{13000}{650 \times 20}}{15} = \frac{7000}{15} = +466.67 = (466.67 \quad 180^\circ)$$

Center of moments at intersection of Q and R

$P \times 15 \cos 26^\circ\, 34' + 600 \times 10 - 650 \times 20 + Q \times 0 + R \times 0 = 0$

$$P = -\frac{\overset{6000}{600 \times 10} - \overset{13000}{650 \times 20}}{\underset{13.41}{15 \times .894}} = \frac{7000}{13.41} = -522.0 = (522.0 \quad 333^\circ 26')$$

Center of moments at intersection of P and R

$Q \times 30 - 600 \times 20 + 650 \times 10 + P \times 0 + R \times 0 = 0$

$$Q = \frac{\overset{2000}{600 \times 20} - \overset{6500}{650 \times 10}}{30} = \frac{5500}{30} = +183.33 = (183.33 \quad 90^\circ)$$

RESULTS

Algebraic	Graphic
$P = (522.0$ lbs. $333^\circ 26')$	$P = (522$ lbs. $330^\circ 26')$
$Q = (183.3$ lbs. $90^\circ\ 0')$	$Q = (183$ lbs. $90^\circ\ 0')$
$R = (466.7$ lbs. $180^\circ\ 0')$	$R = (467$ lbs. $180^\circ\ 0')$

[illegible]

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